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MECHANICS

TRIGONOMETRY

FOR BEGINNERS

WITH NUMEROUS EXAMPLES.

BY

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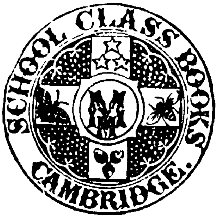
TRIGONOMETRY FOR BEGINNERS.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PREFACE.

THE present work is intended to be placed in the hands of beginners, and to serve as an introduction to the larger treatise on Plane Trigonometry, published by the author. The same plan has been adopted as in the *Algebra for Beginners:* the subject is discussed in short chapters, and a collection of examples is attached to each chapter. Many of these examples are original and have been constructed with reference to the most important points and to the usual difficulties of beginners; the rest have been derived from College and University Examination Papers.

Especial attention has been paid to the numerical calculations which occur in Trigonometry, in order that the work may be suitable for those who wish to confine themselves to the practical solution of triangles, as well as for those who intend to advance in the study of mathematics.

The subject is arranged in the order which appears most convenient for beginners; an acquaintance with the books of Euclid which are usually read, and with the rudiments of Algebra, being all that is assumed. The first fourteen Chapters present the geometrical part of Plane Trigonometry; they contain all that is necessary for practical purposes. The remaining Chapters are of a more analytical character, and are important in the Theory of Mathematics. It will be found that the order of study may be varied at the discretion of the teacher, and the theoretical part taken at an earlier period.

The range of matter included is such as seems required by the various examinations in Elementary Trigonometry which are now carried on in the country; it is hoped that nothing has been omitted which usually finds a place in such examinations.

The Miscellaneous Examples at the end are arranged in sets, each set containing ten examples: the first hundred relate to the first eight Chapters of the book; the second hundred extend to the end of the sixteenth Chapter; and the last hundred relate to the whole book.

Any remarks with respect to the book, and especially the indication of difficulties or omissions in the text or the examples, will be most thankfully received.

I. TODHUNTER.

CAMBRIDGE,
August, 1871.

TRIGONOMETRY FOR BEGINNERS.

I. *Measurement of Angles by Degrees or Grades.*

1. THE word Trigonometry is derived from two Greek words, one signifying a *triangle*, and the other signifying *I measure*. Plane Trigonometry originally denoted the science in which the relations subsisting between the sides and the angles of a plane triangle were investigated, and the modes of investigation were almost entirely geometrical. But now the term Plane Trigonometry has a wider meaning, and comprises investigations with respect to plane angles whether forming a triangle or not, and the investigations are made by the aid of algebraical symbols and formulæ. Before beginning the present treatise the student should therefore become acquainted with Algebra, at least as far as the solution of simple equations. The parts of the elements of Euclid which are usually read are also necessary.

2. We have first to explain how angles are measured. Some angle is selected as the *unit*, and the measure of any other angle is the number of units which it contains. Any angle might be taken for the *unit*, as for example a *right angle;* but a smaller angle than a right angle is found more convenient. Accordingly a right angle is divided into 90 equal parts called *degrees;* and any angle may be estimated by ascertaining the number of degrees which it contains. If the angle does not contain an exact number of degrees we can express it in degrees and a fraction of a degree. A degree is divided into 60 equal parts called *minutes*, and a minute into 60 equal parts called *seconds;* and thus a

fraction of a degree may if we please be converted into minutes and seconds.

3. Thus, for example, half a right angle contains 45 degrees; a quarter of a right angle contains $22\frac{1}{2}$ degrees, which we may write in the decimal notation 22·5 degrees, or we may express it as 22 degrees 30 minutes; one-sixteenth of a right angle contains $5\frac{5}{8}$ degrees, that is, 5·625 degrees, or 5 degrees, 37 minutes, 30 seconds.

4. Symbols are used as abbreviations of the words *degrees*, *minutes*, and *seconds*. Thus $5^{\circ}\ 37'\ 30''$ is used to denote 5 degrees, 37 minutes, 30 seconds.

5. The method of estimating angles by degrees, minutes, and seconds, is almost universally adopted in practical calculations. Another method was proposed in France, towards the end of the last century, in connexion with a uniform system of decimal tables of weights and measures. In this method a right angle is divided into 100 equal parts called *grades*, a grade is divided into 100 equal parts called *minutes*, and a minute is divided into 100 equal parts called *seconds*. On account of the occurrence of the number *one hundred* in forming the subdivisions of a right angle, this method of estimating angles is called the *centesimal* method; and the common method is called the *sexagesimal* method, on account of the occurrence of the number *sixty* in forming the subdivisions of a degree. The centesimal method is also sometimes called the *French* method, and the sexagesimal method is called the *English* method.

6. Symbols are used as abbreviations of the words *grades*, *minutes*, and *seconds*, in the centesimal method. Thus, $5^{g}\ 37^{\backprime}\ 30^{\backprime\backprime}$ is used to denote 5 grades, 37 minutes, 30 seconds in the centesimal method. A centesimal minute and second are not the same as a sexagesimal minute and second, and the accents or dashes which are used to denote minutes and seconds in the two methods are distinguished by sloping in different directions.

7. In the centesimal method any whole number of minutes and seconds may be expressed immediately as a decimal fraction of a grade. Thus, 37 centesimal minutes

are $\frac{37}{100}$ of a grade, that is, ·37 of a grade; and 30 centesimal seconds are $\frac{30}{(100)^2}$ of a grade, that is, ·003 of a grade. Hence $5^g\,37'\,30''$ may be written as $5^g{\cdot}373$; and since a grade is $\left(\frac{1}{100}\right)^{\text{th}}$ of a right angle, $5^g{\cdot}373$ may be written as ·05373 of a right angle. Notwithstanding this great advantage of the centesimal method, the sexagesimal method has been retained in practical calculations, because the latter had become thoroughly established by long use in mathematical works, and especially in mathematical tables, before the former was proposed; and such works and tables would have been rendered almost useless by the change in the method of estimating angles. The centesimal method is not practically used even in France.

8. Although the centesimal method is not used in practical calculations it is customary to give an account of the method in works on Trigonometry; and it is shewn how to compare the numbers which measure the same angle in the English and French methods. This we shall explain in the next three Articles.

9. *To compare the number of degrees in any angle with the number of grades in the same angle.*

Let D be the number of *degrees* in any given angle, G the number of *grades* in the same angle. Then, since there are 90 degrees in a right angle, $\frac{D}{90}$ expresses the ratio of the given angle to a right angle; and, since there are 100 grades in a right angle, $\frac{G}{100}$ also expresses the ratio of the given angle to a right angle.

Hence $$\frac{D}{90}=\frac{G}{100};$$

therefore $$D=\frac{90}{100}G=\frac{9}{10}G=G-\frac{1}{10}G,$$

and $$G=\frac{100}{90}D=\frac{10}{9}D=D+\frac{1}{9}D.$$

The formula $D = G - \frac{1}{10} G$ gives the following rule: *From the number of grades in any angle subtract one-tenth of that number; the remainder is the number of degrees in the angle.*

The formula $G = D + \frac{1}{9} D$ gives the following rule: *To the number of degrees in any angle add one-ninth of that number; the sum is the number of grades in the angle.*

10. *To compare the number of English minutes in any angle with the number of French minutes in the same angle.*

Let m be the number of English minutes in any angle, μ the number of French minutes in the same angle. Then, since there are 90×60 English minutes in a right angle, $\frac{m}{90 \times 60}$ expresses the ratio of the given angle to a right angle; and since there are 100×100 French minutes in a right angle, $\frac{\mu}{100 \times 100}$ also expresses the ratio of the given angle to a right angle.

Hence
$$\frac{m}{90 \times 60} = \frac{\mu}{100 \times 100};$$
therefore
$$m = \frac{9 \times 6}{10 \times 10} \mu = \frac{27}{50} \mu,$$
and
$$\mu = \frac{50}{27} m.$$

11. Similarly, if s be the number of English seconds in any angle, and σ the number of French seconds in the same angle

$$\frac{s}{90 \times 60 \times 60} = \frac{\sigma}{100 \times 100 \times 100};$$
therefore
$$s = \frac{9 \times 6 \times 6}{10 \times 10 \times 10} \sigma = \frac{81}{250} \sigma,$$
and
$$\sigma = \frac{250}{81} s.$$

EXAMPLES. I.

Express the following six angles in the French mode:

1. 54^0.
2. $1^0\,21'$.
3. $6^0\,18'$.
4. $9^0\,49'\,57''$.
5. $27^0\,41'\,51''$.
6. $67^0{\cdot}4325$.

Express the following six angles in the English mode:

7. 30^g.
8. $3^g\,50^{\backprime}$.
9. $10^g\,42^{\backprime}\,50^{\backprime\backprime}$.
10. $20^g\,77^{\backprime}\,50^{\backprime\backprime}$.
11. $31^g\,7^{\backprime}\,50^{\backprime\backprime}$.
12. $76^g{\cdot}452$.

Express the following six angles in both modes:

13. $\frac{1}{5}$ of a right angle.
14. $\frac{5}{8}$ of a right angle.
15. $\frac{11}{16}$ of a right angle.
16. $\frac{45}{64}$ of a right angle.

17. The angle of an equilateral triangle.

18. The angle at the vertex of the isosceles triangle described in Euclid IV. 10.

19. The sum of two angles is 30 grades, and their difference is 9 degrees: find each angle.

20. The difference of the two acute angles of a right-angled triangle is 20 grades: find the angles in degrees.

21. Find the number of English minutes in a grade.

22. Find the number of English seconds in a French minute.

23. Find the number of French minutes in a degree.

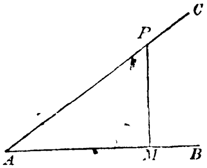
24. Find the number of French seconds in an English minute.

25. Find the ratio of an angle of $1^0\,25'$ to an angle of $1^g\,25^{\backprime}$.

II. *Trigonometrical Ratios.*

12. There are certain quantities connected with an angle which are called the *Trigonometrical Ratios* of the angle; in the present Chapter we shall define these Trigonometrical Ratios, and demonstrate some of their most important properties, confining ourselves to angles less than a right angle. It will be seen as we proceed with the book that the whole subject rests on the definitions and properties contained in the present Chapter.

13. Let BAC be any acute angle; take any point in either of the containing straight lines, and from it draw a perpendicular to the other straight line: let P be the point in AC, and PM perpendicular to AB. We shall use the letter A to denote the angle BAC.



The following are the definitions of the Trigonometrical Ratios of the angle A:

$\frac{PM}{AP}$, that is $\frac{perpendicular}{hypotenuse}$, is called the *sine* of A;

$\frac{AM}{AP}$, that is $\frac{base}{hypotenuse}$, is called the *cosine* of A;

$\frac{PM}{AM}$, that is $\frac{perpendicular}{base}$, is called the *tangent* of A;

$\frac{AM}{PM}$, that is $\frac{base}{perpendicular}$, is called the *cotangent* of A;

$\frac{AP}{AM}$, that is $\frac{hypotenuse}{base}$, is called the *secant* of A;

$\frac{AP}{PM}$, that is $\frac{hypotenuse}{perpendicular}$, is called the *cosecant* of A.

When the cosine of A is subtracted from unity the remainder is called the *versed sine* of A. When the sine of A is subtracted from unity the remainder is called the *coversed* sine of A. But the term *versed sine* is not often used, and the term *coversed sine* is scarcely ever used.

14. The words *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, *versed sine*, and *coversed sine* are usually abbreviated in writing and printing; thus the above definitions may be expressed as follows:

$$\sin A = \frac{PM}{AP}, \qquad \cos A = \frac{AM}{AP},$$

$$\tan A = \frac{PM}{AM}, \qquad \cot A = \frac{AM}{PM},$$

$$\sec A = \frac{AP}{AM}, \qquad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \qquad \operatorname{covers} A = 1 - \sin A.$$

15. The *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, *versed sine*, and *coversed sine* of an angle are called the *Trigonometrical Ratios* of the angle: it will be seen from the definitions that the term *ratio* is appropriate, because each of the quantities defined is the *ratio* of one length to another, that is, each of the quantities is some arithmetical number or fraction. The *Trigonometrical Ratios* have been sometimes called *Trigonometrical Functions*, and sometimes *Goniometrical Ratios* or *Functions*.

16. The excess of a right angle over any angle is called the *complement* of that angle. Thus if A be the number of degrees in any angle, $90 - A$ is the number of degrees in the complement of the angle. This affords another method of defining some of the Trigonometrical Ratios: after defining, as in Art. 13, the *sine*, *tangent*, and *secant* of an angle we may say:

the cosine of an angle is the sine of the complement of that angle;

the cotangent of an angle is the tangent of the complement of that angle;

the cosecant of an angle is the secant of the complement of that angle.

For in the triangle PAM the angle APM is the complement of the angle A; and

$$\sin APM = \frac{perpendicular}{hypotenuse} = \frac{AM}{AP} = \cos A;$$

$$\tan APM = \frac{perpendicular}{base} = \frac{AM}{MP} = \cot A;$$

$$\sec APM = \frac{hypotenuse}{base} = \frac{AP}{MP} = \operatorname{cosec} A.$$

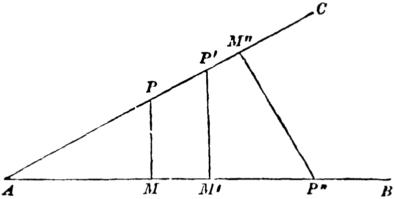
These results may also be expressed thus:

the sine of an angle is the cosine of the complement of that angle;

the tangent of an angle is the cotangent of the complement of that angle;

the secant of an angle is the cosecant of the complement of that angle.

17. *The Trigonometrical Ratios remain unchanged so long as the angle remains unchanged.*



Let BAC be any angle; in AC take any point P, and draw PM perpendicular to AB; also take any other point P', and draw $P'M'$ perpendicular to AB. Then, by similar triangles, Euclid VI. 4,

$$\frac{PM}{AP} = \frac{P'M'}{AP'};$$

that is, the *sine* of the angle A is the same whether it be formed from the triangle APM or from the triangle $AP'M'$.

The same result holds for the other Trigonometrical Ratios.

Or we may suppose a point P'' taken in AB and $P''M''$ drawn perpendicular to AC; then the triangles APM and $AP''M''$ are similar, and

$$\frac{PM}{AP} = \frac{P''M''}{AP''}.$$

18. We have now defined the Trigonometrical Ratios, and have shewn that each Ratio has only one value so long as the angle is unchanged: we proceed to establish certain relations which hold among the Trigonometrical Ratios.

19. We have immediately from the definitions

$$\tan A \times \cot A = \frac{PM}{AM} \times \frac{AM}{PM} = 1,$$

therefore $$\tan A = \frac{1}{\cot A}, \quad \cot A = \frac{1}{\tan A};$$

$$\sec A \times \cos A = \frac{AP}{AM} \times \frac{AM}{AP} = 1,$$

therefore $$\sec A = \frac{1}{\cos A}, \quad \cos A = \frac{1}{\sec A};$$

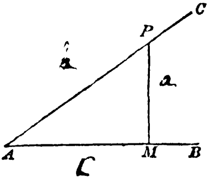
$$\operatorname{cosec} A \times \sin A = \frac{AP}{PM} \times \frac{PM}{AP} = 1,$$

therefore $$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sin A = \frac{1}{\operatorname{cosec} A}.$$

Also $$\tan A = \frac{PM}{AM} = \frac{PM}{AP} \div \frac{AM}{AP} = \frac{\sin A}{\cos A};$$

$$\cot A = \frac{AM}{PM} = \frac{AM}{AP} \div \frac{PM}{AP} = \frac{\cos A}{\sin A}.$$

20. *To shew that* $(\sin A)^2 + (\cos A)^2 = 1$.



In the right-angled triangle APM we have

$$PM^2 + AM^2 = AP^2;$$

therefore $$\frac{PM^2 + AM^2}{AP^2} = 1,$$

therefore $$\frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} = 1,$$

therefore $$\left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = 1;$$

that is, $$(\sin A)^2 + (\cos A)^2 = 1.$$

21. With respect to the preceding demonstration it should be remarked that it is shewn in Euclid I. 47, that the square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the sides; and it is known that the *geometrical* square described on any straight line is measured by the *arithmetical* square of the number which measures the length of the straight line. From combining these two results we obtain the arithmetical equality $PM^2 + AM^2 = AP^2$, which is the foundation of the preceding demonstration.

22. It is usual for shortness to write $(\sin A)^2$ thus, $\sin^2 A$; similarly $(\sin A)^3$ is written thus, $\sin^3 A$. The same mode of abbreviation is used for the powers of the other Trigonometrical Ratios; and so the result obtained in Art. 20 is usually written thus,

$$\sin^2 A + \cos^2 A = 1.$$

23. *To shew that*

$(\sec A)^2 = 1 + (\tan A)^2$, *and* $(\operatorname{cosec} A)^2 = 1 + (\cot A)^2$.

In the right-angled triangle APM we have

$$AP^2 = AM^2 + PM^2,$$

therefore $$\frac{AP^2}{AM^2} = 1 + \frac{PM^2}{AM^2},$$

therefore $$\left(\frac{AP}{AM}\right)^2 = 1 + \left(\frac{PM}{AM}\right)^2;$$

that is, $$(\sec A)^2 = 1 + (\tan A)^2.$$

Again $$AP^2 = PM^2 + AM^2;$$

therefore $$\left(\frac{AP}{PM}\right)^2 = 1 + \left(\frac{AM}{PM}\right)^2;$$

that is, $$(\operatorname{cosec} A)^2 = 1 + (\cot A)^2.$$

The results here obtained are usually written thus,

$$\sec^2 A = 1 + \tan^2 A, \qquad \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

24. By means of the relations which have been established in Arts. 19...23 we can express all the Trigonometrical Ratios in terms of any one of them.

Thus, for example, we will express all the rest in terms of the *sine:*

$$\cos A = \sqrt{(1 - \sin^2 A)} \qquad \text{(Art. 20)},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{(1 - \sin^2 A)}} \quad \text{(Arts. 19, 20)},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{(1 - \sin^2 A)}}{\sin A} \quad \text{(Arts. 19, 20)},$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{(1 - \sin^2 A)}} \quad \text{(Arts. 19, 20)},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} \qquad \text{(Art. 19)},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \sqrt{(1-\sin^2 A)} \quad \text{(Art. 20).}$$

Again, we will express all the rest in terms of the *tangent:*

$$\sin A = \frac{1}{\operatorname{cosec} A} = \frac{1}{\sqrt{(1+\cot^2 A)}} = \frac{1}{\sqrt{\left(1+\dfrac{1}{\tan^2 A}\right)}}$$

$$= \frac{\tan A}{\sqrt{(1+\tan^2 A)}} \quad \text{(Arts. 19, 23),}$$

$$\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{(1+\tan^2 A)}} \quad \text{(Arts. 19, 23),}$$

$$\cot A = \frac{1}{\tan A} \qquad \text{(Art. 19),}$$

$$\sec A = \sqrt{(1+\tan^2 A)} \qquad \text{(Art. 23),}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\sqrt{(1+\tan^2 A)}}{\tan A},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{1}{\sqrt{(1+\tan^2 A)}}.$$

25. If we have given the value of one of the Trigonometrical Ratios we can thus find the values of the rest.

Suppose, for example, that $\sin A = \frac{3}{5}$; then we have

$$\cos A = \sqrt{(1-\sin^2 A)} = \sqrt{\left(1-\frac{9}{25}\right)} = \sqrt{\frac{16}{25}} = \frac{4}{5},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{5} \div \frac{4}{5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4}, \quad \cot A = \frac{1}{\tan A} = \frac{4}{3},$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{4}, \quad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{3},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{4}{5} = \frac{1}{5}.$$

Again, suppose, for example, that $\tan A = \frac{8}{15}$; then we have

$$\sin A = \frac{\tan A}{\sqrt{(1+\tan^2 A)}} = \frac{\frac{8}{15}}{\sqrt{\left(1+\frac{64}{225}\right)}} = \frac{8}{\sqrt{289}} = \frac{8}{17},$$

$$\cos A = \frac{1}{\sqrt{(1+\tan^2 A)}} = \frac{1}{\sqrt{\left(1+\frac{64}{225}\right)}} = \frac{15}{17},$$

$$\cot A = \frac{1}{\tan A} = \frac{15}{8},$$

$$\sec A = \frac{1}{\cos A} = \frac{17}{15}, \qquad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{17}{8},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{15}{17} = \frac{2}{17}.$$

26. By the aid of the ordinary formulæ of Algebra we can deduce from the relations established in Arts. 19...23 various others, which like those are universally true; and which are therefore called *Trigonometrical Identities.*

As an example we can shew that

$$(\sec A + \operatorname{cosec} A)^2 - (\tan A + \cot A)^2 = 2 \sec A \operatorname{cosec} A.$$

For $(\sec A + \operatorname{cosec} A)^2 = \sec^2 A + 2 \sec A \operatorname{cosec} A + \operatorname{cosec}^2 A$,

$$(\tan A + \cot A)^2 = \tan^2 A + 2 \tan A \cot A + \cot^2 A$$

$$= \tan^2 A + 2 + \cot^2 A, \qquad \text{by Art. 19;}$$

therefore $(\sec A + \operatorname{cosec} A)^2 - (\tan A + \cot A)^2$

$$= \sec^2 A - \tan^2 A + \operatorname{cosec}^2 A - \cot^2 A + 2 \sec A \operatorname{cosec} A - 2$$

$$= 1 + 1 + 2 \sec A \operatorname{cosec} A - 2, \qquad \text{by Art. 23,}$$

$$= 2 \sec A \operatorname{cosec} A.$$

EXAMPLES. II.

Find the values of the other Trigonometrical Ratios in the following nine examples, having given:

1. $\sin A = \frac{12}{13}$. 2. $\sin A = \frac{40}{41}$. 3. $\cos A = \cdot 28$.

4. $\cos A = \frac{60}{61}$. 5. $\tan A = \frac{4}{3}$. 6. $\sin A = \frac{1}{3}$.

7. $\cos A = \frac{3}{4}$. 8. $\sin A = \frac{2m}{m^2+1}$. 9. $\cos A = \frac{2mn}{m^2+n^2}$.

Demonstrate the following identities:

10. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.

11. $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$.

12. $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

13. $\sin^4 A + \cos^4 A = 1 - 2\sin^2 A \cos^2 A$.

14. $\tan A + \cot A = \sec A \operatorname{cosec} A$.

15. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.

16. $\sin^2 A \tan A + \cos^2 A \cot A = \frac{1 - 2\sin^2 A \cos^2 A}{\sin A \cos A}$.

17. $\sin^2 A + \operatorname{vers}^2 A = 2(1 - \cos A)$.

18. $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A)$.

19. $\sin^6 A + \cos^6 A = \sin^4 A + \cos^4 A - \sin^2 A \cos^2 A$.

20. $\sin^2 A \tan^2 A + \cos^2 A \cot^2 A = \tan^2 A + \cot^2 A - 1$.

21. $\sin A \tan^2 A + \operatorname{cosec} A \sec^2 A - 2\tan A \sec A$
$= \operatorname{cosec} A - \sin A$.

22. $(\sin A \cos B + \cos A \sin B)^2$
$+ (\cos A \cos B - \sin A \sin B)^2 = 1$

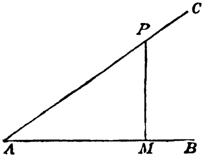
23. $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$.

24. $(1 - \sin A - \cos A)^2 (1 + \sin A + \cos A)^2$
$= 4\sin^2 A$ [illegible]$s^2 A$

25. $(1 + \sin A - \cos A)^2 + (1 + \cos A - \sin A)^2$
$= 4(1 - \sin A$ co[illegible]$)$

III. *Values of the Trigonometrical Ratios for an angle of* 45°, *of* 60°, *of* 30°.

27. *To determine the values of the Trigonometrical Ratios for an angle of* 45°.



Let BAC be an angle of 45°; take any point P in AC, and draw PM perpendicular to AB. Since PAM is half a right angle, APM is also half a right angle; therefore $PM = AM$.

Now $$PM^2 + AM^2 = AP^2;$$

thus $$2PM^2 = AP^2;$$

therefore $$\left(\frac{PM}{AP}\right)^2 = \frac{1}{2}; \text{ therefore } \frac{PM}{AP} = \frac{1}{\surd 2}.$$

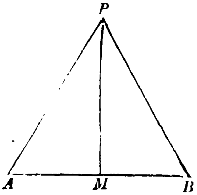
Thus $$\sin 45^0 = \frac{PM}{AP} = \frac{1}{\surd 2}; \ \cos 45^0 = \frac{AM}{AP} = \frac{1}{\surd 2}.$$

$$\tan 45^0 = \frac{PM}{AM} = 1; \ \cot 45^0 = \frac{AM}{PM} = 1.$$

$$\sec 45^0 = \frac{AP}{AM} = \surd 2; \ \operatorname{cosec} 45^0 = \frac{AP}{PM} = \surd 2.$$

$$\operatorname{vers} 45^0 = 1 - \cos 45^0 = 1 - \frac{1}{\surd 2}.$$

28. *To determine the values of the Trigonometrical Ratios for an angle of* 60^0 *and for an angle of* 30^0.



Let APB be an equilateral triangle, so that the angle PAB contains 60 degrees; draw PM perpendicular to AB, then $AM = MB$;

$$\text{therefore } AM = \tfrac{1}{2}AB = \tfrac{1}{2}AP.$$

$$\text{Thus} \quad \cos 60^0 = \frac{AM}{AP} = \frac{1}{2};$$

$$\sin 60^0 = \sqrt{(1 - \cos^2 60)} = \sqrt{\left(1 - \frac{1}{4}\right)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2};$$

$$\tan 60^0 = \frac{\sin 60^0}{\cos 60^0} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}; \qquad \cot 60^0 = \frac{1}{\tan 60^0} = \frac{1}{\sqrt{3}};$$

$$\sec 60^0 = \frac{1}{\cos 60^0} = 2; \qquad \operatorname{cosec} 60^0 = \frac{1}{\sin 60^0} = \frac{2}{\sqrt{3}};$$

$$\operatorname{vers} 60^0 = 1 - \cos 60^0 = 1 - \frac{1}{2} = \frac{1}{2}.$$

The Trigonometrical Ratios for an angle of 30^0 may be found by Art. 16: thus

$$\sin 30^0 = \cos 60^0 = \frac{1}{2}; \qquad \cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2};$$

$$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}; \qquad \cot 30^0 = \tan 60^0 = \sqrt{3};$$

$$\sec 30^0 = \operatorname{cosec} 60^0 = \frac{2}{\sqrt{3}}; \qquad \operatorname{cosec} 30^0 = \sec 60^0 = 2;$$

$$\operatorname{vers} 30^0 = 1 - \cos 30^0 = 1 - \frac{\sqrt{3}}{2}.$$

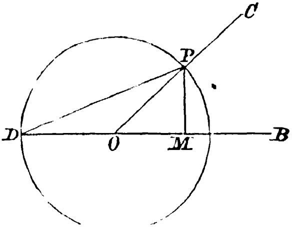
29. We have thus found the values of the Trigonometrical Ratios for an angle of 45°, of 60°, and of 30°. There are other angles for which the values of the Trigonometrical Ratios can also be found, though the expressions for them are less simple than in the cases of the angles which we have considered: for instance, in Chapter XIII. the Trigonometrical Ratios for an angle of 18° and of 36° are found. But there are comparatively few angles for which such expressions can be obtained.

Although the Trigonometrical Ratios are seldom capable of being expressed *exactly*, as they are in the cases of the angles which we have considered, yet they can be found *approximately* for any angle; and the calculations may be carried to any assigned degree of accuracy. We shall not enter into an account of the processes of calculation in the present work, but may refer to the more complete treatise. It will be sufficient to state as a fact that tables may be easily procured which give to seven places of decimals the sine of any angle which can be expressed in degrees and minutes; the other ratios can be determined when the sine is known.

Although we shall not explain the mode in which the tables are constructed, yet the student will readily see as he proceeds with the subject that various formulæ occur which might be useful in calculating the values of the Trigonometrical Ratios. Especially he may notice the formulæ hereafter to be given by which we may determine the Trigonometrical Ratios for an angle which is the sum or the difference of two other angles having known Trigonometrical Ratios. And we shall give a formula in the next Article which will enable us to determine the Trigonometrical Ratios for the *half* of an angle when the Trigonometrical Ratios of the angle itself are known.

In Chapter XVI. we shall consider in detail the range of the values of the Trigonometrical Ratios when the angle changes; but it will be useful to state here some obvious facts. Thus *neither the sine nor the cosine of an angle can ever be greater than unity.* This follows from the definitions of the sine and cosine, since the hypotenuse of a right-angled triangle is greater than either of the sides. In like manner, *neither the secant nor the cosecant of an angle can ever be numerically less than unity*. There are no limits with respect to the magnitude of the tangent and the cotangent; each of these Ratios may be as great as we please or as small as we please by taking suitable values of the angle.

30. *To express the tangent of half an angle in terms of the sine and cosine of the angle.*



Let BOC be any angle, which we will denote by A. Take any point P in OC. With O as centre, and OP as radius, describe a circle. Produce BO to meet the circumference at D. Draw PM perpendicular to OB. Join PD.

By Euclid III. 20 the angle $PDM = \frac{1}{2}A$.

$$\text{Now } \tan PDM = \frac{PM}{DM} = \frac{PM}{DO + OM} = \frac{PM}{OP + OM}.$$

Let $OP = a$; then

$$\frac{PM}{OP} = \sin A, \text{ therefore } PM = a \sin A,$$

$$\frac{OM}{OP} = \cos A, \text{ therefore } OM = a \cos A.$$

$$\text{Thus } \tan PDM = \frac{a \sin A}{a + a \cos A} = \frac{\sin A}{1 + \cos A};$$

that is,
$$\tan \tfrac{1}{2}A = \frac{\sin A}{1 + \cos A}.$$

31. *By the preceding Article when* sin A and cos A are known we can determine tan $\frac{1}{2}A$; and then by Art. 24 we can deduce from tan $\frac{1}{2}A$ the values of the other Trigonometrical Ratios of $\frac{1}{2}A$.

For example, suppose $A = 30^0$, then $\frac{1}{2}A = 15^0$.

$$\tan 15^0 = \frac{\sin 30^0}{1 + \cos 30^0} = \frac{\frac{1}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{1}{2 + \sqrt{3}};$$

we may multiply both numerator and denominator of the last fraction by $2 - \sqrt{3}$, and thus we obtain the more convenient result $\tan 15^0 = 2 - \sqrt{3}$.

By Art. 23 $\sec^2 15^0 = 1 + \tan^2 15 = 1 + (2 - \sqrt{3})^2 = 8 - 4\sqrt{3}$. Hence we find sec 15^0 by taking the square root of $8 - 4\sqrt{3}$; it is shown in Algebra how to extract this square root; it is easy, by squaring both members, to verify that

$$\sqrt{8 - 4\sqrt{3}} = (\sqrt{3} - 1)\sqrt{2}.$$

Hence $$\sec 15^0 = (\sqrt{3} - 1)\sqrt{2};$$

$$\cos 15^0 = \frac{1}{(\sqrt{3} - 1)\sqrt{2}} = \frac{\sqrt{3} + 1}{(\sqrt{3} + 1)(\sqrt{3} - 1)\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}};$$

$$\cot 15^0 = \frac{1}{\tan 15^0} = 2 + \sqrt{3};$$

$$\operatorname{cosec}^2 15^0 = 1 + \cot^2 15 = 8 + 4\sqrt{3},$$

$$\operatorname{cosec} 15^0 = (\sqrt{3} + 1)\sqrt{2};$$

$$\sin 15^0 = \frac{1}{(\sqrt{3} + 1)\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

Since the Trigonometrical Ratios for an angle of 15^0 are known we can immediately deduce those for an angle of 75^0 by Art. 16.

32. The student should render himself perfectly familiar with the values of the Trigonometrical Ratios for an angle of 30^0, 45^0, or 60^0; as they will be perpetually used in the subject. Thus, for example, if an angle of 60^0 occurs it may be necessary to have the cosine of this angle, which

has been found to be $\frac{1}{2}$. And conversely, if the cosine of an angle is known to be $\frac{1}{2}$, and the angle is less than a right angle, the student will immediately infer that the angle contains 60^0. Should there be any difficulty in this inference it will be removed by the remarks made hereafter, in which it will appear why we introduce the restriction that *the angle is less than a right angle.* See Chapter XVIII.

33. It may be observed that if an angle be less than 45^0 the cosine of the angle is greater than the sine, and if an angle be greater than 45^0 and less than 90^0 the cosine is less than the sine: these results follow immediately from the figure in Art. 13, since the greater side in a triangle is opposite to the greater angle.

34. From the result given in Art. 30 we can deduce some other results which will be useful hereafter, while the process will serve to apply some of the formulæ already established.

We have $$(\tan \tfrac{1}{2}A)^2 = \frac{\sin^2 A}{(1+\cos A)^2};$$

hence, by Art. 20,

$$(\tan \tfrac{1}{2}A)^2 = \frac{1-\cos^2 A}{(1+\cos A)^2} = \frac{(1-\cos A)(1+\cos A)}{(1+\cos A)^2}.$$

Thus $$(\tan \tfrac{1}{2}A)^2 = \frac{1-\cos A}{1+\cos A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But, by Art. 23,

$$(\sec \tfrac{1}{2}A)^2 = 1 + (\tan \tfrac{1}{2}A)^2$$

$$= 1 + \frac{1-\cos A}{1+\cos A} = \frac{2}{1+\cos A};$$

therefore, by Art. 19,

$$(\cos \tfrac{1}{2}A)^2 = \frac{1+\cos A}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and $$(\sin \tfrac{1}{2}A)^2 = 1 - (\cos \tfrac{1}{2}A)^2 = 1 - \frac{1+\cos A}{2}$$

$$= \frac{1-\cos A}{2} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

And $\sin A = (1 + \cos A) \tan \frac{1}{2}A$

$$= 2 (\cos \tfrac{1}{2}A)^2 \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \ldots (4).$$

In these Articles we use the form $(\tan \frac{1}{2}A)^2$ as most intelligible for a beginner; but for abbreviation this is commonly written thus, $\tan^2 \frac{1}{2}A$; similarly, $\sin^2 \frac{1}{2}A$ is written for $(\sin \frac{1}{2}A)^2$. See Art. 22.

The results contained in (2) and (3) may be presented thus:

$$\cos A = 2(\cos \tfrac{1}{2}A)^2 - 1 = 1 - 2(\sin \tfrac{1}{2}A)^2 = (\cos \tfrac{1}{2}A)^2 - (\sin \tfrac{1}{2}A)^2.$$

Suppose we put $2B$ for A in these results; thus we obtain

$$\cos 2B = 2(\cos B)^2 - 1 = 1 - 2(\sin B)^2 = (\cos B)^2 - (\sin B)^2 \ldots (5).$$

In like manner suppose we put $2B$ for A in (4); thus we obtain

$$\sin 2B = 2 \sin B \cos B \ldots\ldots\ldots\ldots (6).$$

The student will have to accustom himself to such changes as we have here exemplified; for instance, he must regard (4) and (6) as expressing under slightly different forms exactly the same result, so that from either form the other immediately follows.

We might put the formula into words; and then either (4) or (6) gives us the following enunciation: *twice the product of the sine of an angle into the cosine of the angle is equal to the sine of twice the angle.* But although the verbal statement is in this case sufficiently simple, yet it will not be so in all cases: and the student must learn to see that a formula may have different aspects owing to a difference in the symbols employed, while the fact expressed remains the same.

As another example put $2B$ for A in the formula at the end of Art. 30; thus we obtain

$$\tan B = \frac{\sin 2B}{1 + \cos 2B}.$$

Another demonstration of the results obtained in Art. 30 and in the present Article will be found in Art. 184.

35. We will now illustrate the subject by the solution of a few examples.

(1) Find sin A from the equation $\sin A + \cos A = \frac{7}{5}$.

By transposition we have $\cos A = \frac{7}{5} - \sin A$;

square both sides, thus $\cos^2 A = \frac{49}{25} - \frac{14}{5} \sin A + \sin^2 A$;

therefore $$1 - \sin^2 A = \frac{49}{25} - \frac{14}{5} \sin A + \sin^2 A;$$

therefore $$2 \sin^2 A - \frac{14}{5} \sin A + \frac{49}{25} = 1.$$

This is an ordinary quadratic equation for finding sin A, which we solve in the usual manner; we have

$$\sin^2 A - \frac{7}{5} \sin A = -\frac{12}{25},$$

therefore $$\sin^2 A - \frac{7}{5} \sin A + \left(\frac{7}{10}\right)^2 = \frac{49}{100} - \frac{12}{25} = \frac{1}{100},$$

therefore $\sin A - \frac{7}{10} = \pm \frac{1}{10}$; therefore $\sin A = \frac{4}{5}$ or $\frac{3}{5}$.

(2) Find A and B from the equations

$$\frac{\sin A}{\sin B} = \sqrt{3} \text{...........(1)}, \qquad \frac{\tan A}{\tan B} = 3 \text{...........(2)}.$$

Put $\frac{\sin A}{\cos A}$ for $\tan A$, and $\frac{\sin B}{\cos B}$ for $\tan B$; thus (2) becomes

$$\frac{\sin A}{\sin B} \cdot \frac{\cos B}{\cos A} = 3,$$

therefore by (1) $$\sqrt{3} \frac{\cos B}{\cos A} = 3,$$

therefore $$\frac{\cos B}{\cos A} = \sqrt{3} \text{.................(3)}.$$

From (1) we have $$\sin A = \sqrt{3} \sin B,$$

from (3) we have $\cos A = \frac{1}{\sqrt{3}} \cos B;$

square and add, thus $\sin^2 A + \cos^2 A = 3 \sin^2 B + \frac{1}{3} \cos^2 B,$

therefore $1 = 3 \sin^2 B + \frac{1}{3} (1 - \sin^2 B),$

therefore $3 = 1 + 8 \sin^2 B,$

therefore $\sin^2 B = \frac{1}{4}$, therefore $\sin B = \frac{1}{2}$. Thus $B = 30^0$.

Then from (1) we obtain $\sin A = \frac{\sqrt{3}}{2}$. Thus $A = 60^0$.

(3) Find A and B from the equations

$$\sin(A + B) = \frac{\sqrt{3}}{2}, \quad \tan(A - B) = 1.$$

From the first equation we obtain $A + B = 60^0$; from the second equation we obtain $A - B = 45^0$.

Hence, by addition and subtraction, we have $2A = 105^0$, and $2B = 15^0$; therefore $A = 52\frac{1}{2}^0$, and $B = 7\frac{1}{2}^0$.

(4) Express $\tan 2A$ in terms of $\tan A$.

$$\tan 2A = \frac{\sin 2A}{\cos 2A};$$

substitute for $\cos 2A$ and $\sin 2A$ their values from equations (5) and (6) of Art. 34; thus

$$\tan 2A = \frac{2 \sin A \cos A}{\cos^2 A - \sin^2 A}.$$

Divide both numerator and denominator of the last fraction by $\cos^2 A$, which will not change the value: therefore

$$\tan 2A = \frac{\dfrac{2 \sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}} = \frac{2 \tan A}{1 - \tan^2 A}.$$

Thus $\tan 2A$ is expressed in terms of $\tan A$.

EXAMPLES. III.

Find A from the following equations:

1. $3 \sin A = 2 \cos^2 A$.

2. $\sec A \tan A = 2\sqrt{3}$.

3. $\sec^2 A - \frac{5}{2} \sec A + 1 = 0$.

4. $6 \cot^2 A - 4 \cos^2 A = 1$.

5. $3 \operatorname{cosec}^2 A + 8 \sin^2 A = 10$.

6. $\tan^2 A - 4 \tan A + 1 = 0$.

Find A and B from the following equations:

7. $\frac{\sin A}{\sin B} = \sqrt{2}$, $\frac{\cos A}{\cos B} = \frac{\sqrt{2}}{\sqrt{3}}$.

8. $\frac{\sin A}{\cos B} = \frac{\sqrt{3}}{\sqrt{2}}$, $\frac{\cos A}{\sin B} = \frac{1}{\sqrt{2}}$.

9. $\cos (A - B) = \frac{\sqrt{3}}{2}$, $\sin (A - B) = \cos (A + B)$.

10. $\cos (2A + B) = \frac{1}{2}$, $\sin (3A - B) = \frac{1}{2}$.

11. $\tan (4A + 7B) = 2 + \sqrt{3}$, $\tan (5A - 7B) = 2 - \sqrt{3}$.

12. $\sin A + \sin B = \sqrt{2}$, $\sin^2 A + \sin^2 B = 1$.

13. Find A, B, and C from the equations

$$\cos (A + B + C) = \frac{1}{2},\ \sin (A + B - C) = \frac{1}{2},\ \tan (B + C) = 1.$$

14. Find the Trigonometrical Ratios for an angle of $22\frac{1}{2}^\circ$.

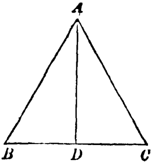
15. Find $\tan 7\frac{1}{2}^\circ$. — 16. Find $\tan 37\frac{1}{2}^\circ$.

IV. *Applications of Trigonometry.*

36. In the present Chapter we shall give some examples of the use of the Trigonometrical Ratios. It will not be possible to supply any great variety or extent of illustration, because at present we have not advanced beyond the simplest elements of the subject; but the student may be led to take more interest in Trigonometry from seeing even at this early stage that it admits of valuable practical applications.

We begin by demonstrating an important proposition, which connects the sides of a triangle with the Trigonometrical Ratios of the angles.

37. *In a triangle the sides are proportional to the sines of the opposite angles.*



Let ABC be a triangle; from A draw AD perpendicular to the opposite side, meeting that side at D.

Thus $\frac{AD}{AB} = \sin B$, therefore $AD = AB \sin B$;

and $\frac{AD}{AC} = \sin C$, therefore $AD = AC \sin C$;

therefore $$AB \sin B = AC \sin C;$$

therefore $$\frac{AB}{AC} = \frac{\sin C}{\sin B}.$$

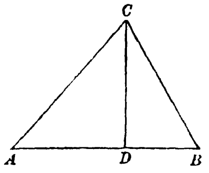
This shews that the proposition is true for any pair of sides.

We suppose that the two angles considered are *acute;* this will be sufficient for the applications we shall make in the present Chapter: the case of a triangle with an *obtuse* angle will be considered hereafter. See Chapter x.

It is usual to denote the lengths of the sides of a triangle opposite to the angles A, B, C respectively by the letters a, b, c.

38. The applications we are about to make of Trigonometry will consist of some examples of the calculation of heights and distances. We shall assume that the lengths of straight lines on the ground can be measured, and also that the angle between any two straight lines which meet at the eye of an observer can be measured. Lengths are usually measured by means of a chain. Angles are usually measured by a sextant or by a theodolite. A sextant will measure the angle between any two straight lines drawn from the observer's eye. A theodolite will measure the angle between any straight line drawn from the observer's eye and the horizontal straight line drawn in the same vertical plane as the former straight line: a theodolite will also measure the angle between two horizontal straight lines drawn from the observer's eye, one in one assigned vertical plane, and the other in another assigned vertical plane. A fuller account of the instruments used in measuring distances will be found in works on Surveying.

39. *To find the distance of an inaccessible point on a horizontal plane.*



Let C be the inaccessible point. Measure any straight line AB in the horizontal plane containing C. At A observe the angle CAB, and at B observe the angle ABC. Then the angle ACB is known, by Euclid I. 32.

Now
$$\frac{AC}{AB} = \frac{\sin ABC}{\sin ACB};$$

therefore
$$AC = \frac{AB \sin ABC}{\sin ACB}:$$

thus AC is known.

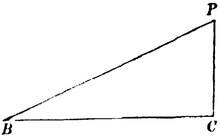
Suppose we require the perpendicular distance CD of C from the straight line AB; we have

$$\frac{CD}{AC} = \sin CAB;$$

therefore
$$CD = AC \sin CAB = \frac{AB \sin ABC \sin CAB}{\sin ACB}:$$

thus CD is known.

40. *To find the height of a visible accessible object.*



Let P be the top of the object, and let it be required to find the height PC. Measure any distance CB in a horizontal straight line from the foot of the object; at B observe the angle PBC. Then

$$\frac{PC}{BC} = \tan PBC,$$

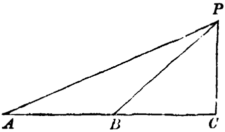
therefore
$$PC = BC \tan PBC:$$

thus PC is known.

41. Strictly speaking, in the preceding Article, BC is not a straight line measured *on* the ground, but a straight line *parallel* to the ground at a distance from it equal to the height of the observer's eye at B. Thus PC is the height of the object above the level of the observer's eye; to obtain the height of P measured from the ground we

must add to the value of PC the height of the observer's eye at B above the ground. This remark will be applicable to some other Articles in the book; we shall not repeat it, nor need the student supply the correction thus noticed unless it should be definitely required in an example.

42. *To find the height and the distance of an inaccessible object on a horizontal plane.*



Let P be the top of an object, and let it be required to find the height PC, and the distance of the object from a given point A in the horizontal plane through C. At A observe the angle PAC, then measure any length AB directly towards the object, and at B observe the angle PBC. Then in the triangle APB the side AB is known, and the angle PAB, and also the angle APB; for the angle APB is the difference of the angles PBC and PAC, by Euclid I. 32.

Now, by Art. 37, $$\frac{BP}{AB} = \frac{\sin PAB}{\sin APB};$$

therefore $$BP = \frac{AB \sin PAB}{\sin APB}.$$

Then $$\frac{PC}{BP} = \sin PBC;$$

therefore $$PC = BP \sin PBC = \frac{AB \sin PAB \sin PBC}{\sin APB}.$$

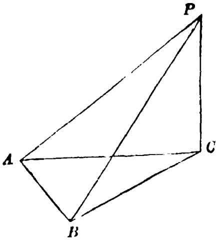
And $$\frac{AC}{PC} = \cot PAC = \frac{\cos PAB}{\sin PAB};$$

therefore $$AC = \frac{AB \cos PAB \sin PBC}{\sin APB}.$$

Thus PC and AC are known.

43. If however it is not convenient to measure the length AB directly towards the object we may proceed thus: measure the length AB in any direction from A; at A observe the angle PAB, and the angle PAC; and at B observe the angle PBA.

Thus PCA and PCB are in two different vertical planes, which intersect in PC.



Then in the triangle ABP the side AB is known, and the angles PAB and PBA; and thus the angle APB is known by Euclid I. 32.

Now $$\frac{AP}{AB}=\frac{\sin ABP}{\sin APB};$$

therefore $$AP=\frac{AB\sin ABP}{\sin APB}.$$

Then $$\frac{PC}{AP}=\sin PAC;$$

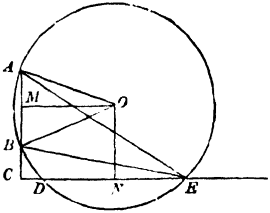
therefore $$PC=AP\sin PAC=\frac{AB\sin PAC\sin ABP}{\sin APB}.$$

And $$\frac{AC}{PC}=\cot PAC=\frac{\cos PAC}{\sin PAC};$$

therefore $$AC=\frac{AB\cos PAC\sin ABP}{\sin APB}.$$

Thus PC and AC are known.

44. *An object of known height is situated above a horizontal plane; and the angle subtended by the object at a given point in the plane is observed: it is required to determine the elevation of the object above the plane.*



Let AB be the object of known height; BC the required elevation above the horizontal plane. Suppose a segment of a circle described on AB, containing an angle equal to the given angle subtended by AB: let this circle cut the horizontal plane at D and E.

Let O be the centre of the circle; draw OM perpendicular to AB, and ON perpendicular to DE.

The angle AOM is half of the angle AOB; hence by Euclid III. 20 the angle AOM is equal to the angle AEB, and is therefore known.

And $$\frac{AM}{MO} = \tan AOM;$$

therefore $$MO = AM \cot AOM;$$

and as AM is half of AB we thus determine MO.

Now one of the two distances CD and CE is supposed to be given, namely, the former or the latter according as the given distance is less or greater than MO; and we can determine the other, since CN is known, for it is equal to MO, and N is the middle point of DE.

Then by Euclid III. 36, *Corollary*,

$$CD \,.\, CE = CA \,.\, CB = (CM + MA)(CM - MA)$$
$$= CM^2 - MA^2.$$

therefore $$CM^2 = CD \,.\, CE + MA^2.$$

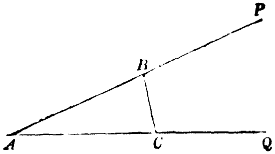
This determines CM, and then CB will be known.

If both the angles AEC and BEC can be observed the problem admits of a much simpler solution; for we have

$$AB = CE\,(\tan AEC - \tan BEC); \quad BC = CE \tan BEC:$$

from the first of these equations we can find CE, and then from the second we can find BC. But there may be cases in which the two angles cannot be conveniently observed; for instance E may be at the foot of a hill on which AB stands, and the position of the horizontal straight line EC may be difficult to fix.

45. It may be remarked that by the aid of measurement of lengths and of calculation we may sometimes avoid the necessity of observing an angle.



Suppose we wish to know the angle PAQ subtended at the point A by straight lines drawn from the points P and Q.

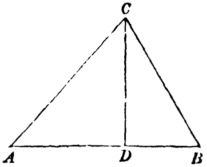
Take any point B in AP; and take C on AQ such that $AC = AB$; and measure BC. Then a perpendicular from A on BC would bisect both the straight line BC and the angle BAC;

therefore $$\sin \tfrac{1}{2} BAC = \frac{\tfrac{1}{2} BC}{AB}.$$

Since the right-hand side of this equation is known, we can find the angle $\tfrac{1}{2}BAC$ by the aid of a Table of Sines; see Art. 29. Thus the angle BAC is determined.

46. We will give one example of the use of Trigonometry in Mensuration. We suppose the student to know that the area of a rectangle is measured by the product of the numbers which represent the lengths of two adjacent sides; see the *Notes on the Second Book of Euclid.* The area of a triangle is therefore represented by half the product of its base and altitude, by Euclid I. 41. We shall now be able to demonstrate the proposition of the following Article.

47. *The area of a triangle is equal to half the produ of two sides into the sine of the included angle.*



Let ABC be any triangle, CD the perpendicular from C on the base AB.

Then $\qquad$ the area $= \frac{1}{2} AB \, . \, CD$;

and $$\frac{CD}{AC} = \sin BAC;$$

therefore $\qquad CD = AC \sin BAC$:

thus the area $= \frac{1}{2} AB \, . \, AC \, . \sin BAC$.

We suppose that the angle A is acute, as this will suffic for our present purpose; but the formula holds also if th angle A be obtuse. This will be seen after the student ha read Chapter IX.

48. *To express the area of a triangle when one si and the angles are known.*

With the figure of the preceding Article we have, as in Art. 39,

$$AC = \frac{AB \sin ABC}{\sin ACB};$$

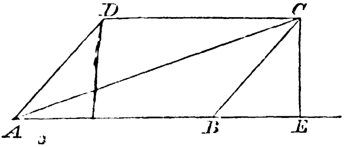
therefore the area of the triangle

$$=\frac{AB^2 . \sin ABC . \sin BAC}{2 \sin ACB}.$$

49. The area of a parallelogram is double that of a triangle having the same base and altitude: hence by Art. 47 the area of a parallelogram is equal to the product of two adjacent sides into the sine of the included angle.

Similarly we may apply Art. 48 to find the area of a parallelogram.

50. *To express the diagonals of a parallelogram in terms of two adjacent sides and the cosine of an acute angle of the figure.*



Let $ABCD$ be the parallelogram; suppose the angle BAD to be acute. From C draw CE perpendicular to AB produced. Then by Euclid II. 12,

$$AC^2 = AB^2 + BC^2 + 2AB . BE.$$

But $$\frac{BE}{BC} = \cos CBE = \cos BAD;$$

thus $$BE = BC \cos BAD; \quad \text{and } BC = AD:$$

therefore $$AC^2 = AB^2 + AD^2 + 2AB . AD \cos BAD.$$

Similarly by drawing a perpendicular from D on AB, and using Euclid II. 13, we find that

$$BD^2 = AB^2 + AD^2 - 2AB . AD \cos BAD.$$

EXAMPLES. IV.

1. Given $\tan A = 1{\cdot}05$, find the other Trigonometrical Ratios.

2. Given $\sin A = \dfrac{2n^2 + 2mn}{2n^2 + 2mn + m^2}$, find the other Trigonometrical Ratios.

3. Given $\tan A = \dfrac{pm + qn}{pn - qm}$, find the other Trigonometrical Ratios.

4. Find $\tan x$ from $\tan x + \cot x = 2a$.

5. Find $\sin x$ from $\sin x + \cos x = a$.

6. Find A from $\sin A + \cos A = \sqrt{\dfrac{3}{2}}$.

7. Find A and B from

$$\tan A \tan B = 1, \quad \tan^2 A + \tan^2 B = \frac{10}{3}.$$

8. Find A and B from

$$\tan A + \tan B = 4, \quad \tan^2 A + \tan^2 B = 14.$$

9. Find A and B from

$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \qquad \sin A \sin B = \frac{1}{4}.$$

10. Shew that the tangents of 60°, 45°, and 15° are in Arithmetical Progression.

11. At a distance of 100 feet from the foot of a tower the tower subtends an angle of 30°: find the height of the tower.

12. A base AB of 100 yards is measured close to the bank of a river, and a tree C on the other bank is observed from A and B; the angle CAB is found to be 60° and the angle CBA is found to be 45°: determine the breadth of the river.

13. A person standing on the bank of a river observes the angle subtended by a tree on the opposite bank to be 75°, and when he retires 20 feet from the bank of the river he observes the angle to be 60°: determine the height of the tree and the breadth of the river.

14. Find the area of an equilateral triangle, each side being equal to a.

15. Find the area of an isosceles triangle, each of the equal sides being equal to a, and the included angle 30°.

16. A man 6 feet high standing at the top of a mast subtends an angle whose tangent is $\frac{1}{10}$ at a point on the deck 33 feet from the foot of the mast: find the height of the mast.

17. The upper half of a post, seen from a point on a level with the foot of the post, subtends an angle whose tangent is $\frac{1}{3}$: find the tangent of the angle subtended by the whole post.

18. A staff at the top of a tower is observed to subtend an angle of 15° by an observer at a distance of a feet from the foot of the tower, and also to subtend the same angle when the observer is at a distance of b feet: find the height of the staff.

19. A column standing on a pedestal 25 feet 6 inches high subtends an angle of 45° at the eye of an observer who stands on the horizontal plane from which the pedestal springs. When the observer approaches 20 feet nearer to the column it again subtends an angle of 45° at his eye. Find the height of the column supposing the height of the observer's eye above the plane to be 5 feet 6 inches.

20. A person wishing to know the height of a wall, the foot of which was inaccessible, fixed an upright staff 5 feet high (the height of his eye), at the place where the angular altitude above the level of his eye was 45°. Having then walked backwards till the angle between the top of the wall and the top of the staff was $18^\circ 26'$, of which the tangent is $\frac{1}{3}$, he found by actual measurement that his distance from the staff was 70 feet. Determine the height of the wall.

V. *Logarithms.*

51. The numerical calculations which occur in the solution of triangles are abbreviated by the aid of logarithms; and thus it is necessary to explain the nature and the properties of logarithms.

Suppose that $a^x = n$, then x is called the *logarithm* of n *to the base* a: thus the logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number.

The logarithm of n to the base a is written $\log_a n$: thus if $a^x = n$, then $x = \log_a n$.

52. For example $4^3 = 64$, so that 3 is the logarithm of 64 to the base 4; or $\log_4 64 = 3$.

Again, required the logarithm of 27 to the base 9. Let x denote the required logarithm, so that $9^x = 27$: thus $(3^2)^x = 3^3$, that is $3^{2x} = 3^3$; therefore $2x = 3$, that is $x = 1\frac{1}{2}$.

In the next three Articles we shall give the properties on which the utility of logarithms chiefly depends.

53. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

For let $x = \log_a m$, and $y = \log_a n$;

therefore $m = a^x$, and $n = a^y$;

therefore $mn = a^{x+y}$;

therefore $\log_a mn = x + y = \log_a m + \log_a n$.

54. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For let $x = \log_a m$, and $y = \log_a n$;

therefore $m = a^x$, and $n = a^y$;

therefore $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$;

therefore $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$

55. *The logarithm of any power, integral or fractional, of a number is equal to the product of the logarithm of the number and the index of the power.*

For let $m = a^x$; therefore $m^r = (a^x)^r = a^{rx}$,
therefore $\log_a (m^r) = rx = r \log_a m.$

56. *To find the relation between the logarithms of the same number to different bases.*

Let $x = \log_a m$, and $y = \log_b m$;
therefore $m = a^x$ and $= b^y$;
therefore $a^x = b^y$;
therefore $a^{\frac{x}{y}} = b$, and $b^{\frac{y}{x}} = a$;
therefore $\frac{x}{y} = \log_a b$, and $\frac{y}{x} = \log_b a.$

Hence $y = x \log_b a$, and $= \frac{x}{\log_a b}.$

Hence the logarithm of a number to the base b may be found by multiplying the logarithm of the number to the base a by $\log_b a$ or by $\frac{1}{\log_a b}$.

Since $\log_b a = \frac{1}{\log_a b}$ we have $\log_b a \times \log_a b = 1.$

57. There are two systems of logarithms which are used in Mathematics.

In one system the base is a certain number which cannot be expressed exactly; as far as nine places of decimals the number is 2·718281828.

This number is usually denoted by the letter e; and logarithms to this base are called *Napierian logarithms*, from Napier the inventor of logarithms. This system of logarithms, although very important in theory, is not used in practical calculations; and we shall not require to consider it in the present work.

In the other system the base is 10; this system is used in practical calculations, and is called the *common* system.

58. We shall not in the present work explain how a table of logarithms is calculated; for this the student may refer to the larger treatise. We may remark that in very few cases can a logarithm be assigned *exactly*, but as close an approximate value as we please can be found; for example, a table may be constructed which shall give logarithms to seven places of decimals.

We shall shew in the next three Articles what are the chief advantages of the common system of logarithms.

59. *In the common system of logarithms if the logarithm of any number be known, we can immediately determine the logarithm of the product or quotient of that number by any power of* 10.

For $\log_{10}(N \times 10^n) = \log_{10} N + \log_{10} 10^n = \log_{10} N + n$;

$$\log_{10}\frac{N}{10^n} = \log_{10} N - \log_{10} 10^n = \log_{10} N - n.$$

That is, if we know the logarithm of any number we can determine the logarithm of any number which has the same figures, but differs merely by the position of the decimal point.

In future we shall for brevity use *log* for log_{10}, that is we shall omit to specify the base 10.

60. We know from Arithmetic that

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \ldots$$

Now from this we infer that if a number lies between 1 and 10, its logarithm lies between 0 and 1; if a number lies between 10 and 100, its logarithm lies between 1 and 2; if a number lies between 100 and 1000, its logarithm lies between 2 and 3: and so on.

For example, the logarithm of 3·27 lies between 0 and 1; the logarithm of 74·584 lies between 1 and 2; the logarithm of 659·45 lies between 2 and 3; and so on.

The integral part of a logarithm is called the *characteristic*, and the decimal part is called the *mantissa;* thus as the logarithm of any number between 100 and 1000 is greater than 2 and less than 3, it is equal to 2 + some decimal: thus in this case 2 is the characteristic.

We shall now give an important proposition respecting the characteristic.

61. *In the common system of logarithms the characteristic of the logarithm of any number can be determined by inspection.*

For suppose the number to be greater than unity, and to lie between 10^n and 10^{n+1}; then the logarithm is greater than n and less than $n+1$, so that the characteristic of the logarithm is n. Next suppose the number to be less than unity, and to lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is between 10^{-n} and 10^{-n-1}; then the logarithm will be some negative quantity between $-n$ and $-(n+1)$; hence if we agree that the *mantissa shall always be positive*, the characteristic of the logarithm will be $-(n+1)$.

Hence we have the following rule: the characteristic of the logarithm of a number is *one less* than the number of integral figures of the number; when the number has no integral figures the characteristic of the logarithm is negative and is *one more* than the number of cyphers immediately to the right of the decimal place in the number.

62. By reason of the properties explained in the three preceding Articles it is unnecessary in a table of common logarithms to print either the characteristics of the logarithms or the decimal points of the numbers.

For example, we find in a table the following figures:

Number.	Logarithm.
15627	1938756

This means that ·1938756 is the *mantissa;* for the number 15627 the corresponding characteristic is 4, and therefore log 15627 = 4·1938756. Similarly log 156·27 = 2·1938756, and log ·0015627 = $\bar{3}$·1938756: in the last example $\bar{3}$ is equivalent to -3, so that we express in the manner indicated the fact that log ·0015627 = -3 + ·1938756.

63. It is necessary to notice one point in practical operations with negative characteristics.

Suppose we require the logarithm of the cube root of ·0015627. By Art. 55 the logarithm is $\frac{1}{3}$ of $\bar{3}\cdot1938756$. The division here can be immediately effected; for $\frac{1}{3}$ of -3 is -1; and $\frac{1}{3}$ of ·1938756 is ·0646252: thus the required logarithm is $\bar{1}\cdot0646252$.

But suppose we require the logarithm of the square root of ·0015627. By Art. 55 the logarithm is $\frac{1}{2}$ of $\bar{3}\cdot1938756$. It is convenient now to put $\bar{3}\cdot1938756$ in the form $-4+1\cdot1938756$; then dividing by 2 we obtain $-2+\cdot5969378$, so that the required logarithm is $\bar{2}\cdot5969378$.

Similarly if we require the logarithm of the sixth root of ·0015627 we put $\bar{3}\cdot1938756$ in the form $-6+3\cdot1938756$; then dividing by 6 we obtain $-1+\cdot5323126$, so that the required logarithm is $\bar{1}\cdot5323126$.

64. The following examples will illustrate the present Chapter.

(1) Find the logarithm of 125 to the base 25.

Let x denote the required logarithm; then $25^x=125$; therefore $(5^2)^x=5^3$; therefore $5^{2x}=5^3$; therefore $2x=3$; therefore $x=\frac{3}{2}$.

(2) Having given the logarithms of 72 and 75, find the logarithms of 2 and 3.

Denote the logarithm of 72 by a, and the logarithm of 75 by b.

Then $a=\log 72=\log(8\times9)=\log(2^3\times3^2)$;

therefore $a=3\log 2+2\log 3$..................................(1).

And $b=\log 75=\log(3\times25)=\log\frac{3\times100}{4}=\log\frac{3\times100}{2^2}$;

therefore $b=\log 3+\log 100-2\log 2=\log 3+2-2\log 2$;

therefore $b-2=\log 3-2\log 2$..............................(2).

From (1) and (2) we can find log 2 and log 3 by the usual process for solving simultaneous simple equations. We shall obtain

$$\log 2 = \frac{1}{7}(a - 2b + 4), \qquad \log 3 = \frac{1}{7}(2a + 3b - 6).$$

(3) Find x from the equation $(1{\cdot}6)^x = 2$, having given $\log 2 = {\cdot}3010300$.

Take the logarithms of the two members of the equation;

thus $$x \log \frac{16}{10} = \log 2,$$

therefore $$x(\log 16 - \log 10) = \log 2,$$

therefore $$x(4 \log 2 - 1) = \log 2,$$

therefore $$x = \frac{\log 2}{4 \log 2 - 1} = \frac{{\cdot}30103}{{\cdot}20412} = 1{\cdot}4748 \text{ nearly}.$$

(4) Given $\log 3 = {\cdot}4771213$ find the log of $\dfrac{(2{\cdot}7)^3 \times ({\cdot}81)^{\frac{4}{5}}}{(90)^{\frac{5}{4}}}$.

Let N denote the given expression; then

$$\log N = \log (2{\cdot}7)^3 + \log ({\cdot}81)^{\frac{4}{5}} - \log (90)^{\frac{5}{4}}$$

$$= 3 \log 2{\cdot}7 + \frac{4}{5} \log {\cdot}81 - \frac{5}{4} \log 90.$$

Now $$\log 2{\cdot}7 = \log \frac{27}{10} = \log \frac{3^3}{10} = 3 \log 3 - 1,$$

$$\log {\cdot}81 = \log \frac{81}{100} = \log \frac{3^4}{10^2} = 4 \log 3 - 2,$$

$$\log 90 = \log 3^2 \times 10 = 2 \log 3 + 1;$$

hence $\log N =$

$$3(3 \log 3 - 1) + \frac{4}{5}(4 \log 3 - 2) - \frac{5}{4}(2 \log 3 + 1)$$

$$= \left(9 + \frac{16}{5} - \frac{5}{2}\right) \log 3 - 3 - \frac{8}{5} - \frac{5}{4}$$

$$= \frac{97}{10} \log 3 - \frac{117}{20} = \bar{2}{\cdot}7780766 \text{ nearly}.$$

EXAMPLES. V.

Find the logarithms of the following six numbers to the assigned bases:

1. 256 to the base 2. 2. 32 to the base 4.

3. 243 to the base 9. 4. 16 to the base 8.

5. 64 to the base 16. 6. 128 to the base 32.

Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, find the logarithms of the following twelve numbers:

7. 18. 8. 60. 9. 216. 10. 6480.

11. 5400. 12. $\frac{4}{9}$. 13. 4·32. 14. ·72.

15. ·375. 16. ·03. 17. $6^{-\frac{1}{3}}$. 18. $(5\frac{1}{3})^{-\frac{1}{2}}$.

Given $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$, find the logarithms of the following three numbers:

19. 63. 20. $\frac{9}{49}$. 21. $\frac{1}{441}$.

Given $\log 8 = \cdot 9030900$, $\log 9 = \cdot 9542425$, find the logarithms of the following three numbers:

22. $1\frac{1}{8}$. 23. $\sqrt{1\frac{1}{2}}$. 24. $\sqrt[3]{1\frac{1}{5}}$.

25. Given $\log 8 = \cdot 9030900$, $\log 27 = 1\cdot 4313638$, find $\log 2\frac{1}{4}$.

26. Write down the characteristics of the logarithms of 3·4512, 345·12, ·034512, and ·000034512: also having given $\log \cdot 34512 = \bar{1}\cdot 5379701$, find the logarithm of the product of the above four numbers.

27. The decimal part of the log of 36541 is ·5627804, find the log of $\sqrt[15]{(\cdot 000036541)}$.

28. Find the log of ·0625 to the base 8.

29. Given $\log 1\cdot 4 = \cdot 1461280$, $\log 1\cdot 5 = \cdot 1760913$, find log ·000315.

30. Given log 2 find the log of 1000 to the base 50.

31. Given log 2 find the log of 50 to the base 25.

32. Given log 2 and log 3 find x from $(1\cdot 08)^x = 1000$.

VI. *Use of Tables.*

65. Many collections of Mathematical Tables have been published, differing in extent and in the number of decimal places to which they are carried; and thus practical calculators are enabled to provide themselves with such Tables as are most convenient for the special work on which they may be engaged. A collection of Tables published by W. and R. Chambers may suffice for ordinary purposes. A cheap and very extensive collection of Tables has been edited in Germany by Schrön, and this work has been introduced into England with a Preface by Professor De Morgan.

66. Collections of Tables usually contain explanations of the mode in which they are arranged, together with instructions for using them. We shall accordingly only give here some examples which will suffice to guide the student who may wish to use any Tables for occasional calculation. We shall not give investigations of the accuracy of the methods which we exemplify; for such investigations the student is referred to the larger treatise.

67. One general consideration which applies to the use of Mathematical Tables is this: we rarely find what we require immediately in the Tables, but we find two entries between which what we require must lie, and from which it must be determined. Accordingly we have to exemplify the method of proceeding in such cases.

68. *To find the logarithm of a given number.*

If the given number is contained in the Table we take the decimal part of the logarithm from the Table, and prefix the characteristic; see Art. 61.

Suppose however that the number is not contained exactly in the Table. The Table, for example, may give the logarithms of all numbers from 1 to 100000, and we may require the logarithm of 5632147.

Here we take from the table

Number.	Logarithm.
56321	7506704
56322	7506781.

Hence, by Art. 61,

$$\log 5632100 = 6{\cdot}7506704,$$
$$\log 5632200 = 6{\cdot}7506781.$$

The difference of the two numbers is 100, and the difference of the two logarithms is ·0000077. Let x denote the quantity to be added to the logarithm of 5632100, in order to produce the logarithm of 5632147; then *we assume* that

$$100 : 47 :: {\cdot}0000077 : x;$$

that is, we assume that for a small change in the number there is a *proportional* small change in the logarithm.

Hence we obtain $x = \frac{47}{100} \times {\cdot}0000077$, that is ·000003619, or to seven places of decimals ·0000036.

And $\qquad {\cdot}7506704 + {\cdot}0000036 = {\cdot}7506740.$

Therefore $\qquad \log 5632147 = 6{\cdot}7506740.$

Then, by Art. 61, we can immediately express the logarithm of any other number which is formed from 5632147 by supplying a decimal point; for example

$$\log 56321{\cdot}47 = 4{\cdot}7506740.$$

69. *To find the number corresponding to a given logarithm.*

If the decimal part of the given logarithm is contained in the Table we take the corresponding number, and put a decimal point in the number in the place indicated by the given characteristic.

Suppose however that the decimal part of the logarithm is not contained exactly in the Table; we shall then have to perform a process like that exemplified in the preceding Article. For example, suppose the given logarithm to be 2·7506740.

As before we have

$$\log 5632100 = 6{\cdot}7506704,$$
$$\log 5632200 = 6{\cdot}7506781.$$

Let x denote the quantity to be added to 5632100 to produce the number which has 6·7506740 for its logarithm.

Since $6{\cdot}7506781 - 6{\cdot}7506704 = {\cdot}0000077$,

and $6{\cdot}7506740 - 6{\cdot}7506704 = {\cdot}0000036$,

we form the proportion

$$ {\cdot}0000077 : {\cdot}0000036 :: 100 : x. $$

Hence $x = \dfrac{3600}{77} = 47$ approximately,

therefore $\log 5632147 = 6{\cdot}7506740$,

and therefore $\log 563{\cdot}2147 = 2{\cdot}7506740$.

Thus the required number is 563·2147.

70. In using *Trigonometrical* Tables processes have to be performed like those exemplified in the two preceding Articles for Tables of *Logarithms*. For example, we may have a Table of the sines of those angles which are expressible exactly in degrees and minutes, and we may require the sine of such an angle as $20^\circ\, 14'\, 20''$: in this case we must proceed as in Art. 68.

We take from the Table

$$ \sin 20^\circ\, 14' = {\cdot}3458441. $$

$$ \sin 20^\circ\, 15' = {\cdot}3461171. $$

Let x denote the quantity to be added to ·3458441 to produce the sine of $20^\circ\, 14'\, 20''$.

Since ${\cdot}3461171 - {\cdot}3458441 = {\cdot}0002730$,

we form the proportion

$$ 60'' : 20'' :: {\cdot}0002730 : x. $$

Hence $x = \dfrac{20}{60} \times {\cdot}0002730 = {\cdot}0000910$;

therefore $\sin 20^\circ\, 14'\, 20'' = {\cdot}3458441 + {\cdot}0000910 = {\cdot}3459351$.

71. Tables such as those referred to in the preceding Article are called Tables of *Natural* sines, cosines, tangents,...to distinguish them from other Tables which are called Tables of *Logarithmic* sines, cosines, tangents,... We shall now consider the latter kind of Tables.

72. The Trigonometrical Ratios of an angle are numerical quantities, and it is found very convenient to have Tables which give the *logarithms* of these numerical quantities, so that we may be saved the trouble of calculating them by the aid of the Table of Logarithms. For abbreviation log sin A is used to denote the logarithm of the sine of A; and in a similar manner log cos A, log tan A,... are used.

73. Since the sine of an angle is never greater than unity the logarithm of a sine will never be a positive quantity; the same remark applies to the cosine. In order to avoid the occurrence of negative quantities in the Tables it is found convenient to *add 10 to the logarithm of every Trigonometrical Ratio* before registering it in the Tables; the logarithm so increased is called the *Tabular logarithm*, and is usually denoted by the letter L. Thus

$$L \sin A = \log \sin A + 10,$$
$$L \tan A = \log \tan A + 10,$$

and so on. Of course in calculation we shall have to remember and allow for this addition to the real logarithms of the Trigonometrical Ratios.

74. There is one point to which *special attention must be paid* in using both the Tables of the natural Trigonometrical Ratios and the Tables of the logarithmic Trigonometrical Ratios, namely, that as the angle *increases* the sine, tangent, and secant *increase*, but the cosine, cotangent and cosecant *decrease*; the bearing of this remark will be illustrated in the next Article.

75. We will now give some examples of the use of the Tables of the logarithmic Trigonometrical Ratios.

Given $L \sin 20^\circ\, 14' = 9{\cdot}5388804$,

$L \sin 20^\circ\, 15' = 9{\cdot}5392230$;

required $L \sin 20^\circ\, 14'\, 20''$.

The difference of the given Tabular logarithms is ·0003426, which corresponds to the difference 60″ in the angles; so we form the proportion

$$60 : 20 :: {\cdot}0003426 : x.$$

Hence $$x = \frac{20}{60} \times {\cdot}0003426 = {\cdot}0001142\,;$$

therefore $$L \sin 20^0\, 14'\, 20'' = 9{\cdot}5388804 + {\cdot}0001142$$
$$= 9{\cdot}5389946.$$

The same data will furnish an example of the calculation of a logarithmic cosine.

Given $$L \cos 69^0\, 45' = 9{\cdot}5392230,$$
$$L \cos 69^0\, 46' = 9{\cdot}5388804\,;$$

required $$L \cos 69^0\, 45'\, 40''.$$

Here the proportion is

$$60 : 40 :: {\cdot}0003426 : x.$$

Hence $$x = \frac{40}{60} \times {\cdot}0003426 = {\cdot}0002284\,;$$

therefore $$L \cos 69^0\, 45'\, 40'' = 9{\cdot}5392230 - {\cdot}0002284$$
$$= 9{\cdot}5389946.$$

Here x is *subtracted* from $L \cos 69^0\, 45'$ because as the angle increases, the cosine decreases, and so also does the L cosine.

The two preceding examples resemble that in Art. 68; we will now take one resembling that in Art. 69.

Given $$L \sin 20^0\, 14' = 9{\cdot}5388804,$$
$$L \sin 20^0\, 15' = 9{\cdot}5392230\,;$$

find the angle which has for its L sine $9{\cdot}5389946$.

Let x denote the required number of seconds.

Since $$9{\cdot}5392230 - 9{\cdot}5388804 = {\cdot}0003426,$$

and $$9{\cdot}5389946 - 9{\cdot}5388804 = {\cdot}0001142,$$

we form the proportion

$${\cdot}0003426 : {\cdot}0001142 :: 60 : x.$$

Hence $$x = 60 \times \frac{{\cdot}0001142}{{\cdot}0003426} = 20.$$

Hence the required angle is $20^0\, 14'\, 20''$.

EXAMPLES. VI.

1. Given log 34586 = 4·5389003, log 34587 = 4·5389129, find log 345·8637.

2. Given log 41501 = 4·6180586, log 41502 = 4·6180690, find log 4150145.

3. Given log 7·3510 = ·8663464, log 7·3511 = ·8663523, find log 735·1092.

4. Given log 1752 = 3·2435341, log 1752·1 = 3·2435589, find log 17·52087.

5. Given log 6·1025 = ·7855078, log 6·1026 = ·7855149, find log 610·257.

6. Given log 61875 = 4·7915152, log 61876 = 4·7915222, find log 6187539.

7. Given log 61601 = 4·7895878, log 61602 = 4·7895948, find the number corresponding to the logarithm 2·7895912.

8. Given log 7·5014 = ·8751423, log 7·5015 = ·8751481, find the number corresponding to the logarithm $\bar{3}$·8751462.

9. Given log 1·3107 = ·1175033, log 131·08 = 2·1175364, log 5 = ·6989700, find the seventeenth root of 131072.

10. Find $(1·05)^{20}$, having given

log 2 = ·3010300, log 2·653 = ·4237372,
log 3 = ·4771213, log 2·654 = ·4239009,
log 7 = ·8450980.

11. Find L sin $38^0\,24'\,27''$, having given

L sin $38^0\,24' = 9·7931949$,
L sin $38^0\,25' = 9·7933543$.

12. Find L sin $32^0\,28'\,36''$, having given

L sin $32^0\,28' = 9·7298197$,
L sin $32^0\,29' = 9·7300182$.

13. Find L sin $41^0\,50'\,34''·5$, having given

L sin $41^0\,50'\,30'' = 9·8241743$,
L sin $41^0\,50'\,40'' = 9·8241978$.

14. Find $L \cos 17^\circ 31' 25''\cdot 2$, having given
$L \cos 17^\circ 31' = 9\cdot 9793796$,
$L \cos 17^\circ 32' = 9\cdot 9793398$.

15. Find $L \tan 21^\circ 17' 12''$, having given
$L \tan 21^\circ 17' = 9\cdot 5905617$,
$L \tan 21^\circ 18' = 9\cdot 5909351$.

16. Find $L \tan 27^\circ 26' 42''$, having given
$L \tan 27^\circ 26' = 9\cdot 7152419$,
$L \tan 27^\circ 27' = 9\cdot 7155508$.

17. Find $L \tan 55^\circ 37' 53''$, having given
$L \tan 55^\circ 37' = 10\cdot 1647616$,
diff. for $1' = \cdot 0002711$.

18. Find $L \operatorname{cosec} 33^\circ 10' 20''$, having given
$L \sin 33^\circ 10' = 9\cdot 7380479$,
$L \sin 33^\circ 11' = 9\cdot 7382412$.

19. Given $L \sin 16^\circ = 9\cdot 4403381$, diff. for $1' = \cdot 0004403$,
$L \cos 16^\circ = 9\cdot 9828416$, diff. for $1' = \cdot 0000362$,
find $L \sec 16^\circ 0' 27''$ and $L \tan 16^\circ 0' 27''$.

20. Find A, having given
$L \sin A = 9\cdot 4488105$,
$L \sin 16^\circ 19' = 9\cdot 4486227$,
$L \sin 16^\circ 20' = 9\cdot 4490540$.

21. Find A, having given
$L \sin A = 9\cdot 0787743$,
$L \sin 6^\circ 53' = 9\cdot 0786310$,
$L \sin 6^\circ 53' 10'' = 9\cdot 0788054$.

22. Find A, having given
$L \cos A = 9\cdot 9657056$,
$L \cos 22^\circ 28' 20'' = 9\cdot 9657025$,
$L \cos 22^\circ 28' 10'' = 9\cdot 9657112$.

23. Find A, having given
$L \cos A = 9\cdot 2000000$,
$L \cos 80^\circ 53' = 9\cdot 1998793$,
$L \cos 80^\circ 52' 50'' = 9\cdot 2000105$.

24. If $\log x = \sqrt[3]{10}$ find x, having given
$\log 21544 = 4\cdot 3333263$, $\log 14270 = 4\cdot 1544240$,
$\log 21545 = 4\cdot 3333465$, $\log 14271 = 4\cdot 1544544$.

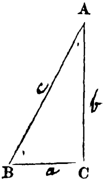
VII. *Solution of Right-Angled Triangles.*

76. In every triangle there are six elements, namely the three sides and the three angles. When we have a sufficient number of these elements given we can calculate the remaining elements; this process is called the *solution of triangles.* It will appear as we proceed that when three of the elements are given we can calculate the remaining three, except when the three angles are given, and then we cannot determine the three sides but only the *ratio* they bear to each other.

We have already in Chapter IV. given some examples of the solution of triangles, and in the course of the present Chapter and a future Chapter we shall examine every case which can occur. It is usual to consider separately the case of right-angled triangles as the investigations are more simple for these than for other triangles; accordingly we shall confine ourselves in the present Chapter to right-angled triangles.

77. We shall use the notation given in Art. 37 for the sides of a triangle; and we shall always suppose that C is the right angle in a right-angled triangle. We use *log* as an abbreviation for logarithm, and we use L in the sense explained in Art. 73.

78. *To solve a right-angled triangle having given the hypotenuse and an acute angle.*



Suppose the hypotenuse and the angle A given; then

$$B = 90^0 - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } a = c \sin A,$$

therefore $\log a = \log c + \log \sin A = \log c + L \sin A - 10;$

$$\frac{b}{c} = \sin B, \text{ therefore } b = c \sin B,$$

therefore $\log b = \log c + \log \sin B = \log c + L \sin B - 10.$
Thus B, a, and b are determined.

79. *To solve a right-angled triangle having given the hypotenuse and a side.*

Suppose c and a given; then

$$\sin A = \frac{a}{c}, \text{ therefore } \log \sin A = \log a - \log c,$$

therefore $$L \sin A = 10 + \log a - \log c;$$

this determines A; then $B = 90^0 - A$.

And $c^2 = a^2 + b^2$, therefore $b^2 = c^2 - a^2 = (c-a)(c+a)$,

therefore $$b = \sqrt{(c-a)(c+a)},$$

$$\log b = \tfrac{1}{2} \log (c-a) + \tfrac{1}{2} \log (c+a).$$

Or we may find b from the formula

$$b = c \cos A.$$

80. *To solve a right-angled triangle having given a side and an acute angle.*

Suppose a and A given; then

$$B = 90^0 - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } c = \frac{a}{\sin A},$$

therefore $\log c = \log a - \log \sin A = \log a - L \sin A + 10;$

$$\frac{a}{b} = \tan A, \quad \text{therefore } b = \frac{a}{\tan A},$$

therefore $\log b = \log a - \log \tan A = \log a - L \tan A + 10.$
Thus B, c, b are determined.

Suppose a and B are given; then $A = 90^0 - B$, and we may find c and b as before.

81. *To solve a right-angled triangle having given the two sides.*

Here a and b are given; then

$$\tan A = \frac{a}{b}, \text{ therefore } \log \tan A = \log a - \log b,$$

therefore $$L \tan A = 10 + \log a - \log b;$$

$$B = 90^\circ - A;$$

$$\frac{a}{c} = \sin A, \text{ therefore } c = \frac{a}{\sin A};$$

therefore $$\log c = \log a - \log \sin A = \log a - L \sin A + 10.$$
Or we may find c from the formula $c = \sqrt{(a^2 + b^2)}$, but this is not adapted to logarithmic computation.

82. It will thus be seen that in each of the four cases discussed in Arts. 78 to 81 we suppose that we know two elements of a right-angled triangle, besides the right angle, and we shew how the other elements are to be determined. And it will be found on examination that we have discussed every case in which two elements are given besides the right angle, except the case in which the two angles are given. In this case we can determine the *ratio* of each side to the hypotenuse; for we have

$$\frac{a}{c} = \sin A, \qquad \frac{b}{c} = \sin B;$$

but we cannot absolutely determine a, b, and c. We may observe that since $A + B = 90^\circ$ it is superfluous to give the values of both A and B, for if one is given the other can be immediately found.

83. We will now take some examples of the solution of right-angled triangles.

Example (1). Suppose we have given $c = 125$, $A = 54^\circ 28'$.

$$a = c \sin A.$$

Using a table of natural sines we find

$$\sin 54^\circ 28' = \cdot 8137775;$$

therefore $$a = 101 \cdot 7221875.$$

Or, using logarithms,

$$\begin{aligned} \log 125 &= 2{\cdot}0969100 \\ L \sin 54^\circ 28' &= 9{\cdot}9105057 \\ &\overline{12{\cdot}0074157} \end{aligned}$$

therefore $\log a = 2{\cdot}0074157$.

Now on consulting the tables of logarithms we find

$$\log 101{\cdot}73 = 2{\cdot}0074490,$$
$$\log 101{\cdot}72 = 2{\cdot}0074064;$$

and as $\log a$ lies between the logarithms here given we conclude that a lies between $101{\cdot}72$ and $101{\cdot}73$.

In order to determine a more closely we must employ the *Principle of proportional parts* which is explained in Chapter VI. We will give the process for the present case.

$$\begin{array}{rlcrl} & 2{\cdot}0074490 & \quad & & 2{\cdot}0074157 \\ & \underline{2{\cdot}0074064} & & & \underline{2{\cdot}0074064} \\ \text{Diff.} & {\cdot}0000426 & & \text{Diff.} & {\cdot}0000093 \end{array}$$

Let x denote the excess of a above $101{\cdot}72$; then

$$\cdot 0000426 : \cdot 0000093 :: \cdot 01 : x.$$

Hence we find $x = {\cdot}00218$ nearly.

Thus $$a = 101{\cdot}72 + {\cdot}00218 = 101{\cdot}72218.$$

Our two modes of calculation give values for a which differ very slightly.

$$B = 90^\circ - A = 35^\circ 32',$$
$$b = c \sin B.$$

Using a table of natural sines we find

$$\sin 35^\circ 32' = {\cdot}5811765,$$
$$b = 72{\cdot}6470625.$$

Or, using logarithms,

$$\begin{aligned} \log 125 &= 2{\cdot}0969100 \\ L \sin 35^\circ 32' &= 9{\cdot}7643080 \\ &\overline{11{\cdot}8612180} \end{aligned}$$

therefore $$\log b = 1{\cdot}8612180.$$

Now on consulting the tables of logarithms we find

$$\log 72{\cdot}648 = 1{\cdot}8612237,$$
$$\log 72{\cdot}647 = 1{\cdot}8612177.$$

Hence b is very nearly equal to $72{\cdot}647$.

Example (2). Suppose we have given $a = 147$, $c = 184$.

$$L \sin A = 10 + \log a - \log c.$$
$$\log 147 = 2{\cdot}1673173,$$
$$\log 184 = 2{\cdot}2648178,$$

therefore $$L \sin A = 9{\cdot}9024995.$$

Now on consulting the tables we find

$$L \sin 53^0\, 2' = 9{\cdot}9025389,$$
$$L \sin 53^0\, 1' = 9{\cdot}9024438\,;$$

and we conclude that A lies between $53^0\, 1'$ and $53^0\, 2'$. Let x denote the number of seconds in the excess of A above $53^0\, 1'$. Proceed as before,

$$\begin{array}{rrrr} & 9{\cdot}9025389 & & 9{\cdot}9024995 \\ & 9{\cdot}9024438 & & 9{\cdot}9024438 \\ \hline \text{Diff.} & {\cdot}0000951 & \text{Diff.} & {\cdot}0000557 \end{array}$$

$${\cdot}0000951 : {\cdot}0000557 :: 60 : x.$$

Thus $x = 35$ nearly,

and $A = 53^0\, 1'\, 35''$ nearly,

$$B = 90^0 - A = 36^0\, 58'\, 25''.$$

If we had only a table of natural sines we should proceed thus:

$$\sin A = \frac{a}{c} = \frac{147}{184} = {\cdot}7989130 \text{ nearly.}$$

On consulting the tables we find

$$\sin 53^0\, 2' = {\cdot}7989855,$$
$$\sin 53^0\, 1' = {\cdot}7988105\,;$$

and we conclude that A lies between $53^0\, 1'$ and $53^0\, 2'$. Then we may proceed as before to determine A more closely; and we shall obtain $A = 53^0\, 1'\, 35''$ nearly.

And $b^2 = c^2 - a^2 = (c-a)(c+a) = 37 \times 331$,

$$\begin{array}{r} \log 37 = 1{\cdot}5682017 \\ \log 331 = 2{\cdot}5198280 \\ \hline 2\,|\,4{\cdot}0880297 \\ \hline \log b = 2{\cdot}0440148 \end{array}$$

Now on consulting the tables of logarithms we find

$$\log 110{\cdot}67 = 2{\cdot}0440299,$$
$$\log 110{\cdot}66 = 2{\cdot}0439907;$$

and hence b lies between 110·66 and 110·67.

Let x denote the excess of b above 110·66. Proceed as before;

$$\begin{array}{rr} & 2{\cdot}0440299 \\ & 2{\cdot}0439907 \\ \hline \text{Diff.} & {\cdot}0000392 \end{array} \qquad \begin{array}{rr} & 2{\cdot}0440148 \\ & 2{\cdot}0439907 \\ \hline \text{Diff.} & {\cdot}0000241 \end{array}$$

$${\cdot}0000392 : {\cdot}0000241 :: {\cdot}01 : x.$$

Thus $\qquad x = {\cdot}00615$ nearly;

and $\qquad b = 110{\cdot}66615$ nearly.

We might also obtain b by extracting the square root of 37×331; without using logarithms.

Example (3). Suppose we have given $a = 237{\cdot}6$, $A = 34^0\,18'$.

$$\log c = \log a - L \sin A + 10,$$

$$\begin{array}{r} \log 237{\cdot}6 = 2{\cdot}3758464 \\ L \sin 34^0\,18' = 9{\cdot}7509140 \\ \hline \log c = 2{\cdot}6249324 \end{array}$$

Now on consulting the tables of logarithms we find

$$\log 421{\cdot}64 = 2{\cdot}6249418,$$
$$\log 421{\cdot}63 = 2{\cdot}6249315;$$

and hence c lies between 421·63 and 421·64.

Let x denote the excess of c above 421·63. Proceed as before;

$$\begin{array}{rr@{\qquad}rr} & 2{\cdot}6249418 & & 2{\cdot}6249324 \\ & 2{\cdot}6249315 & & 2{\cdot}6249315 \\ \cline{2-2}\cline{4-4} \text{Diff.} & {\cdot}0000103 & \text{Diff.} & {\cdot}0000009 \end{array}$$

$$\cdot 0000103 : \cdot 0000009 :: \cdot 01 : x.$$

Thus $$x = \cdot 0009 \text{ nearly};$$

and $$c = 421{\cdot}6309 \text{ nearly}.$$

$$\log b = \log a - L \tan A + 10,$$

$$\begin{array}{rl} \log 237{\cdot}6 = & 2{\cdot}3758464 \\ L \tan 34^\circ\, 18' = & 9{\cdot}8338823 \\ \cline{2-2} \log b = & 2{\cdot}5419641 \end{array}$$

Now on consulting the tables of logarithms we find

$$\log 348{\cdot}31 = 2{\cdot}5419659,$$

$$\log 348{\cdot}30 = 2{\cdot}5419535;$$

and hence b lies between 348·30 and 348·31.

Let x denote the excess of b above 348·30. Proceed as before,

$$\begin{array}{r@{\qquad}r} 2{\cdot}5419659 & 2{\cdot}5419641 \\ 2{\cdot}5419535 & 2{\cdot}5419535 \\ \hline {\cdot}0000124 & {\cdot}0000106 \end{array}$$

$$\cdot 0000124 : \cdot 0000106 :: \cdot 01 : x.$$

Thus $$x = \cdot 00855 \text{ nearly};$$

and $$b = 348{\cdot}30855 \text{ nearly}.$$

84. The process of applying the principle of proportional parts is in practice much simplified by the aid of the Tables; we find in fact that the chief part of the calculation is performed for us. This will be obvious to the student on examining the Tables and the explanations which usually accompany them.

EXAMPLES. VII.

Solve the following ten triangles from the given quantities:

1. $c = 150$, $A = 30^\circ$, $C = 90^\circ$.

2. $c = 200$, $a = 100$, $C = 90^\circ$.

3. $a = 80$, $B = 15^\circ$, $C = 90^\circ$.

4. $a = 75$, $b = 75$, $C = 90^\circ$.

5. $c = 120$, $B = 36^\circ$, $C = 90^\circ$.
$\sin 36^\circ = \cdot 5877853$, $\sin 54^\circ = \cdot 8090170$.

6. $c = 25$, $a = 7$, $C = 90^\circ$.
$\sin 16^\circ\, 15'\, 37'' = \cdot 28$.

7. $c = 290$, $a = 200$, $C = 90^\circ$.
$\sin 43^\circ\, 36' = \cdot 6896195$, $\sin 43^\circ\, 37' = \cdot 6898302$.

8. $a = 125$, $B = 22\frac{1}{2}^\circ$, $C = 90^\circ$.

9. $a = 21$, $b = 20$, $C = 90^\circ$.
$\tan 46^\circ\, 23'\, 50'' = 1{\cdot}05$.

10. $a = 3$, $b = 4$, $C = 90^\circ$.
$\log 2 = \cdot 3010300$, $L \sin 53^\circ\, 7' = 9{\cdot}9030136$,
$L \sin 53^\circ\, 8' = 9{\cdot}9031084$.

In solving the following four triangles the Tables will be required:

11. $c = 196$, $A = 23^\circ\, 30'$, $C = 90^\circ$.

12. $c = 164$, $a = 96$, $C = 90^\circ$.

13. $a = 124{\cdot}6$, $A = 64^\circ\, 20'$, $C = 90^\circ$.

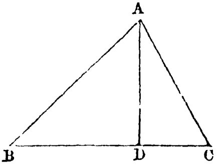
14. $a = 141$, $b = 193$, $C = 90^\circ$.

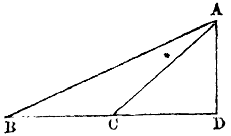
VIII. *Solution of oblique-angled triangles by the aid of right-angled triangles.*

85. In the preceding Chapters we have restricted ourselves to the Trigonometrical Ratios of angles not greater than a right angle; and we have in effect given a short course of Trigonometry as far as the solution of right-angled triangles inclusive. It is however obvious that we may have triangles with obtuse angles, and this leads us to extend our definitions of the Trigonometrical Ratios so as to include angles greater than a right angle. Accordingly we shall devote the next Chapter to this subject; and then we shall proceed in the following Chapters to explain certain properties of triangles and the general solution of triangles.

But it may be convenient for some students to be able to solve any triangle without entering on the consideration of the Trigonometrical Ratios of angles greater than a right angle; and the present Chapter will supply the necessary rules and explanations. Those who adopt the methods of solution to be given in Chapter XI. may look on the present Chapter as an application and illustration of the elementary formulæ of the subject.

86. *To solve a triangle having given two angles and a side.*





Suppose c the given side; since two angles are given all the angles are known. Draw AD perpendicular to BC or to BC produced.

In the right-angled triangle *ABD* we know the hypotenuse *AB*, and the angle *ABD*; hence we can find *AD* and *BD*, by Art. 78.

Then in the right-angled triangle *ADC* we know *AD* and the angle *ACD*; hence we can find *AC* and *CD*, by Art. 80.

And, knowing *BD* and *CD*, we find *BC* immediately.

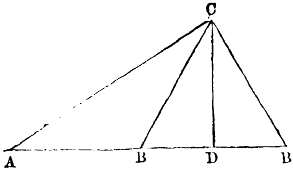
87. *To solve a triangle having given two sides and the included angle.*

The solution is given in Art. 104, and the first part of Art. 110, and as it requires no principles which have not been already explained it may be read at this stage.

88. *To solve a triangle having given two sides and the angle opposite to one of them.*

Let *a* and *b* be the given sides, and *A* the given angle.

I. Suppose *a* less than *b*.



Then *A* is less than *B*, and so *A* must be an acute angle.

Let *AC* denote the side *b*, and from *C* draw the perpendicular *CD* on the opposite side of the triangle, produced if necessary.

In the right-angled triangle *ACD* we know *AC* and the angle *A*; hence we can find *CD* and *AD*, and the angle *ACD*, by Art. 78.

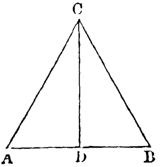
Then in the right-angled triangle *CBD* we know *CD* and *CB*; hence we can find *BD* and the angle *BCD*, by Art. 79.

And thus *AB* and the angle *ACB* may be found immediately.

It will be seen that there are *two* triangles ABC, one having the angle ABC *obtuse*, and one having the angle ABC *acute*. Hence this is usually called the *ambiguous case*, because corresponding to the given elements *two* triangles may *generally* be found.

We say that two triangles may *generally* be found; there will *not always* be two triangles. For it may happen that CD is equal to the given quantity a; and then the two points marked B in the figure will both coincide with D, and there will be only *one* triangle with the given elements, namely the right-angled triangle ACD. Again, it may happen that CD is greater than the given quantity a, and then there is no triangle with the given elements.

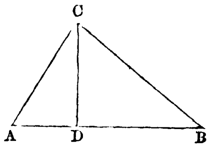
II. Suppose a equal to b.

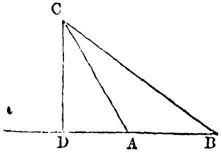


In this case the triangle is isosceles: we have $A = B$.

Then in the right-angled triangle ACD we know AC and the angle A; hence we can find AD. And AB is twice AD.

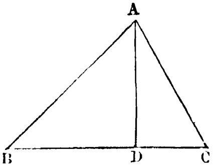
III. Suppose a greater than b.





Then as in I. we solve first the right-angled triangle ACD, and next the right-angled triangle BCD. The given angle A may be acute or obtuse; but there is only one triangle corresponding to the given elements. For if we were to take a point B' to the left of D, on BA produced, such that $DB'=DB$, we should have $CB'=CB=a$, but the angle CAB' of the triangle CAB' would not be equal to the given angle A: in the left-hand figure the angle A is acute while the angle CAB' would be obtuse, and in the right-hand figure the angle A is obtuse while the angle CAB' would be acute.

89. *To solve a triangle having given the three sides.*



Let a denote the side which is less than neither of the others, so that the angles B and C must be acute. Draw AD perpendicular to BC.

Let $BD=x$; then $DC=a-x$.

Now $AD^2=AB^2-BD^2=AC^2-DC^2$;

thus $c^2-x^2=b^2-(a-x)^2=b^2-a^2+2ax-x^2$,

therefore $$x=\frac{c^2+a^2-b^2}{2a}.$$

Thus x is determined.

Then in the right-angled triangle ABD we know AB and BD; hence we can find the angle B. And in the right-angled triangle ACD we know AC and CD; hence we can find the angle C.

In fact we have

$$\cos B=\frac{x}{c}, \text{ and } \cos C=\frac{a-x}{b}.$$

EXAMPLES. VIII.

Solve the following six triangles from the given quantities:

1. $c=84$, $B=45^\circ$, $C=30^\circ$.
2. $b=96$, $c=48$, $A=60^\circ$.
3. $a=1$, $b=1+\sqrt{3}$, $A=15^\circ$.
4. $a=\sqrt{2}$, $b=2(\sqrt{3}-1)$, $A=75^\circ$.
5. $a=\sqrt{3}$, $b=1$, $A=120^\circ$.
6. $a=10$, $b=5\sqrt{3}$, $c=5$.

In solving the following six triangles the Tables will be required:

7. $c=124{\cdot}5$, $A=56^\circ\ 15'$, $B=48^\circ\ 35'$.
8. $a=b=275{\cdot}6$, $C=72^\circ\ 40'$.
9. $a=156$, $b=218$, $A=36^\circ$.
10. $a=750$, $b=625$, $A=80^\circ$.
11. $a=360$, $b=288$, $A=125^\circ$.
12. $a=125$, $b=170$, $c=200$.

13. In the *ambiguous case* when a, b, and A are given, shew that $c=b\cos A\pm\sqrt{(a^2-b^2\sin^2 A)}$.

14. If the perpendicular drawn from the vertex of a triangle on the base fall within the triangle, shew that the difference of the segments of the base is to the difference of the sides as the sum of the sides is to the base.

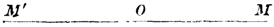
Shew how to solve a triangle having given the two sides and the difference of the segments into which the base is divided by the perpendicular from the vertex.

15. The sides of a triangle are 68, 75 and 77: find the length of the perpendicular on the largest side from the opposite angle.

IX. *Application of Algebraical Signs.*

90. In the preceding Chapters we have defined the Trigonometrical Ratios and established certain relations between them; and we have illustrated the use of the Trigonometrical Ratios. We have hitherto confined ourselves to angles not exceeding a right angle, but it is obvious that angles greater than a right angle may occur in mathematical investigations and in practice; and it becomes necessary to consider how the Trigonometrical Ratios apply to such angles.

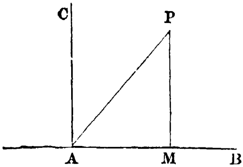
91. Let O be a fixed point in a fixed straight line, and suppose we have to determine the positions of other points in this straight line with respect to O. The position of any point in the straight line will be known if we know the distance of the point from O, and also know *on which side of O the point lies.* Now it is found convenient to adopt the following *convention:* distances measured in one direction from O along the fixed straight line are denoted by *positive* numbers, and distances measured in the opposite direction from O along the fixed straight line are denoted by *negative* numbers. Thus, for example, suppose that distances measured from O towards the *right hand* are denoted by positive numbers, and let M be a point the distance of which from O is denoted by 2 or $+2$; then if M' be as far from O as M is, and on the other side of O, the distance of M' from O is denoted by -2.

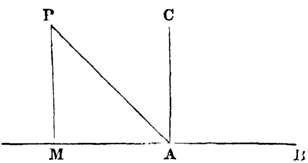


92. We have called this method of determining position by means of numbers affected with algebraical signs a *convention;* we mean by this word to indicate that it is not absolutely *necessary* to adopt this method, but merely *convenient.* The symbols + and − are defined in the

beginning of elementary works on Algebra as indicative of the *operations* of addition and subtraction respectively. As the student advances in Algebra he finds that the symbols + and − are also used as indicative of the *qualities* of quantities. And it is seen that no contradiction or confusion ultimately arises from this double mode of considering the symbols, but that Algebra gains thereby considerably in power. (See *Algebra*, Chaps. v. and xiv.)

93. We shall now extend our definitions of the Trigonometrical Ratios so as to make them applicable to any angle not greater than two right angles.





Let AB, AC be two straight lines at right angles; let a straight line turn round the point A from AB towards AC, and come into any position AP: draw PM perpendicular to AB or to AB produced through A. Then consider AP as always positive; consider AM as positive or negative according as M is on the same side of AC as B is, or on the opposite side; PM will in all cases be on the same side of AB as C is, and will be considered as positive. Let the angle PAB be denoted by A: then the Trigonometrical Ratios of A are thus defined,

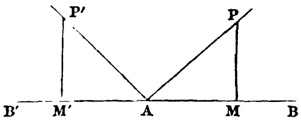
$$\sin A = \frac{PM}{AP}, \quad \tan A = \frac{PM}{AM}, \quad \sec A = \frac{AP}{AM},$$

$$\cos A = \frac{AM}{AP}, \quad \cot A = \frac{AM}{PM}, \quad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \quad \operatorname{covers} A = 1 - \sin A.$$

94. The excess of two right angles over any angle is called the *supplement* of that angle. Thus if A be the number of degrees in any angle, $180 - A$ is the number of degrees in the supplement of the angle.

95. *To compare the Trigonometrical Ratios of any angle with those of the supplement.*



Let PAB be any angle; produce BA to B', and make $P'AB' = PAB$: take $AP' = AP$ and draw PM and $P'M'$ perpendicular to BB'.

The angle $P'AB = 180^\circ - P'AB' = 180^\circ - PAB$; thus $P'AB$ is the supplement of PAB. The triangles PAM and $P'AM'$ are geometrically equal in all respects. Now, by definition,

$$\sin A = \frac{PM}{AP}, \ \sin(180^\circ - A) = \frac{P'M'}{AP'};$$

and since PM and $P'M'$ are equal in magnitude and both positive, we have

$$\sin A = \sin(180^\circ - A).$$

Also, by definition,

$$\cos A = \frac{AM}{AP}, \ \cos(180^\circ - A) = \frac{AM'}{AP'}.$$

Now AM and AM' are equal in magnitude, but since they are measured in opposite directions from A, they are of *opposite signs:*

thus $$\cos A = -\cos(180^\circ - A).$$

The other Trigonometrical Ratios of the angle A may be compared with those of the supplement either by direct use of the figure, or by employing the two results already established; thus, adopting the latter method,

$$\tan(180^0 - A) = \frac{\sin(180^0 - A)}{\cos(180^0 - A)} = \frac{\sin A}{-\cos A} = -\tan A,$$

$$\cot(180^0 - A) = \frac{\cos(180^0 - A)}{\sin(180^0 - A)} = \frac{-\cos A}{\sin A} = -\cot A,$$

$$\sec(180^0 - A) = \frac{1}{\cos(180^0 - A)} = \frac{1}{-\cos A} = -\sec A,$$

$$\operatorname{cosec}(180^0 - A) = \frac{1}{\sin(180^0 - A)} = \frac{1}{\sin A} = \operatorname{cosec} A,$$

$$\operatorname{vers}(180^0 - A) = 1 - \cos(180^0 - A) = 1 + \cos A.$$

Hence the sine and cosecant of any angle are respectively the same as the sine and cosecant of the supplement of the angle; the cosine, tangent, cotangent, and secant of any angle are *numerically* equal to the corresponding Ratios of the supplement of the angle, but are of opposite sign.

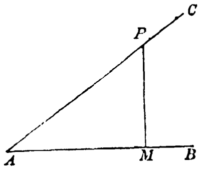
96. It follows from the preceding Article that if the sine of an angle be given, and we have to determine the angle without exceeding two right angles, there will be in general *two* angles which may be taken. For example, if the sine of an angle be $\frac{1}{2}$, we know by Art. 28 that 30^0 is one value of the corresponding angle; and by Art. 95 we have $180^0 - 30^0$, that is 150^0, for another value. Similar remarks apply for the cosecant. But for the other Trigonometrical Ratios this does not hold.

97. It is essential to have a distinct conception of the limits to which the values of the Trigonometrical Ratios tend, when the angle becomes very small.

The straight line AP is supposed to turn round A, remaining always of the same length.

First take the sine,

$$\sin PAB = \frac{PM}{AP}.$$



Now the figure shews that the smaller the angle PAB is the smaller PM is, that is the smaller is the sine; and by taking PAB small enough the sine will become as small as we please. These results are abbreviated thus:

the sine of 0^0 *is* 0.

Next take the cosine,

$$\cos PAB = \frac{AM}{AP}.$$

Now the figure shews that the smaller the angle PAB is the nearer is AM to AP in magnitude, that is, the nearer is the cosine to unity; and by taking PAB small enough the cosine will approach as near to unity as we please. These results are abbreviated thus:

the cosine of 0^0 *is* 1.

Next take the tangent,

$$\tan PAB = \frac{PM}{AM}.$$

Now the figure shows that the smaller the angle PAB is the smaller PM is and the nearer AM is to AP, that is, the smaller is the tangent; and by taking PAB small enough the tangent will become as small as we please. These results are abbreviated thus:

the tangent of 0^0 *is* 0.

Next take the cotangent,

$$\cot PAB = \frac{AM}{PM}.$$

Now as we have already stated, the smaller the angle PAB is the smaller PM is and the nearer AM is to AP, that is, the larger the cotangent is; and by taking PAB small enough the cotangent will become as great as we please. These results are abbreviated thus:

the cotangent of 0^0 *is* ∞.

Next take the cosecant,

$$\operatorname{cosec} PAB = \frac{AP}{PM}.$$

reasoning is similar to that in the case of the cotangent, and the results may be abbreviated thus:

the cosecant of 0^0 *is* ∞.

Next take the secant,

$$\sec PAB = \frac{AP}{AM}.$$

The reasoning is similar to that in the case of the cosine, and the results may be abbreviated thus:

the secant of 0^0 *is* 1.

Since vers $PAB = 1 - \cos PAB$ we have results which we may abbreviate thus:

the versed sine of 0^0 *is* 0.

98. We have obtained the previous results by repeated reference to the figure, but it should be observed that from the first two results the others might have been inferred. Thus, for example,

since $$\tan PAB = \frac{\sin PAB}{\cos PAB},$$

the tangent of 0^0 may be said to be equal to $\frac{0}{1}$, that is to 0.

Again $$\cot PAB = \frac{\cos PAB}{\sin PAB};$$

thus the cotangent of 0^0 may be said to be equal to $\frac{1}{0}$, that is to infinity.

99. It is also necessary to have a distinct conception of the limits to which the Trigonometrical Ratios tend when the angle becomes very nearly a right angle. These may be obtained from the figure, in the manner of Art. 97; or they may be deduced from the results given in Art. 97. We shall adopt the latter method.

Thus, by Art. 16,

$$\begin{aligned}
\sin 90^0 &= \cos 0^0 = 1,\\
\cos 90^0 &= \sin 0^0 = 0,\\
\tan 90^0 &= \cot 0^0 = \infty,\\
\cot 90^0 &= \tan 0^0 = 0,\\
\sec 90^0 &= \operatorname{cosec} 0^0 = \infty,\\
\operatorname{cosec} 90^0 &= \sec 0^0 = 1,\\
\operatorname{vers} 90^0 &= 1 - \cos 90^0 = 1,
\end{aligned}$$

100. Finally it is necessary to have a distinct conception of the limits to which the Trigonometrical Ratios tend when the angle becomes very nearly two right angles. These also may be obtained from the figure in the manner of Art. 97; or they may be deduced by the aid of Art. 95, from the results given in Art. 97. We shall adopt the latter method.

Thus,

$$\begin{aligned}
\sin 180^0 &= \sin 0^0 = 0,\\
\cos 180^0 &= -\cos 0^0 = -1,\\
\tan 180^0 &= -\tan 0^0 = 0,\\
\cot 180^0 &= -\cot 0^0 = \infty,\\
\sec 180^0 &= -\sec 0^0 = -1,\\
\operatorname{cosec} 180^0 &= \operatorname{cosec} 0^0 = \infty,\\
\operatorname{vers} 180^0 &= 1 - \cos 180^0 = 2.
\end{aligned}$$

101. One remark may be made to prevent a possible misconception of some of the preceding results. We have put ∞ as the value of tan 90^0. The student must not assume that ∞ means necessarily $+\infty$. If an angle is a little less than 90^0, the tangent is large and *positive;* if the angle is a little greater than 90^0, the tangent is large and *negative;* so that in saying that tan 90^0 is ∞ we must not suppose that the sign $+$ is necessarily to be taken before ∞. So also in other cases. We have put ∞ as the value of cot 180^0, whereas the student might have expected $-\infty$. It is true that if an angle is a little less than 180^0, the cotangent is negative; but if, as in a subsequent Chapter, we suppose an angle a little greater than 180^0, the cotangent is positive.

The following table collects many of the results obtained in the present Chapter and Chapter III.

	0^0	30^0	45^0	60^0	90^0	120^0	135^0	150^0	180^0
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

EXAMPLES. IX.

1. Find the number of degrees in each angle of a regular pentagon. Find also the number of grades.

2. The number of grades in an angle is equal to two-thirds of the number of degrees in the supplement of the angle: determine the angle.

3. There are as many degrees in the complement of A as in the supplement of $4A$: determine A.

4. Given $\sin A = \frac{5}{13}$, find the Trigonometrical Ratios for $\frac{1}{2}A$.

5. Given $\cot A = 2 - \sqrt{3}$, shew that $\sec A = \sqrt{6} + \sqrt{2}$, and $\operatorname{cosec} A = \sqrt{6} - \sqrt{2}$.

6. If $\tan A = 1 + \sqrt{2}$, find $\cos^2 A$, $\sin^2 A$, and $\cos 2A$.

7. If $\tan A \tan B = 1$, shew that $\sec A = \operatorname{cosec} B$.

8. If $\tan 2A = -\frac{3}{4}$, find $\cos 2A$, $\sin 2A$, $\cos A$, and $\sin A$.

9. Shew that $\tan 52\frac{1}{2}^\circ = (\sqrt{3} + \sqrt{2})(\sqrt{2} - 1)$.

10. Given $\sin A = m \sin B$, and $\cos A = n \cos B$, find $\sin^2 A$ and $\sin^2 B$.

11. Given $\sin A = m \sin B$, and $\tan A = n \tan B$, find $\sin^2 A$ and $\sin^2 B$.

12. If $\tan A + \sec A = \frac{3}{2}$, shew that $\sin A = \frac{5}{13}$.

13. If $\tan A + \sec A = a$, shew that $\sin A = \frac{a^2 - 1}{a^2 + 1}$.

14. Given $\sin A + \operatorname{cosec} A = 2\frac{1}{1640}$, find $\sin A$.

15. If $\sin A + \operatorname{cosec} A = a$, shew that $\sin A = \frac{a - \surd(a^2 - 4)}{2}$.

16. If $$a \cos^2 x + b \sin^2 x = m \cos^2 y,$$
and $$a \sin^2 x + b \cos^2 x = n \sin^2 y,$$
find $\sin^2 x$ and $\sin^2 y$.

17. Shew that $\frac{\tan A}{\tan A - \tan B} + \frac{\cot A}{\cot A - \cot B} = 1$.

18. Shew that $(4\cos^2 A - 1)^2 \tan^2 A + (3 - 4\cos^2 A)^2 = \sec^2 A$.

19. Shew geometrically that $\sin 2A$ is less than $2 \sin A$.

20. Shew that
$$\operatorname{cosec} A (\sec A - 1) + \sin A = \cot A (1 - \cos A) + \tan A.$$

21. Given $a = 5$, $b = 20$, $C = 90^\circ$, find A and B.
$$\log 5 = \cdot 6989700,\ L \tan 75^\circ 57' = 10 \cdot 6016170,$$
$$L \tan 75^\circ 58' = 10 \cdot 6021537.$$

22. Given $a = 9 \cdot 65$, $b = 12 \cdot 24$, $C = 90^\circ$, find A and B:
$$\log 2 = \cdot 3010300,\ \log 153 = 2 \cdot 1846914,\ \log 193 = 2 \cdot 2855573,$$
$$L \tan 38^\circ 15' = 9 \cdot 8967116,\ L \tan 38^\circ 16' = 9 \cdot 8969714.$$

23. If $\sin A + \cos A = a + \surd(1 - a^2)$, shew that either $\sin A$ or $\cos A$ is equal to a.

24. If $\sin A = a + \sqrt{\left(\frac{1}{2} - a^2\right)}$, find $\cos A$.

25. Shew that the theorem of Art. 30 holds if the angle A is greater than one right angle, and not greater than two right angles.

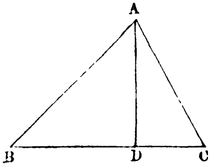
26. Let a and b denote adjacent sides of a parallelogram, γ the angle which they include, and c the diagonal drawn from this angle to the opposite angle: shew that $c^2 = a^2 + b^2 + 2ab \cos \gamma$.

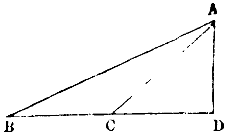
X. *Properties of Triangles.*

102. The present Chapter will contain some properties of triangles which are useful for the solution of triangles. We begin with a proposition which we have already given in Art. 37, but which must be repeated in order to shew that it holds for oblique-angled triangles as well as for acute-angled triangles.

We retain the notation of Art. 37.

103. *In any triangle the sides are proportional to the sines of the opposite angles.*





Let ABC be a triangle, and from A draw AD perpendicular to the opposite side, meeting that side, or that side produced, at D.

If B and C are *acute* angles, we have from the left-hand figure

$$AD = AB \sin B, \text{ and } AD = AC \sin C;$$

therefore $$AB \sin B = AC \sin C,$$

therefore $$\frac{AB}{AC} = \frac{\sin C}{\sin B},$$

that is $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be *obtuse*, we have from the right-hand figure

$$\begin{aligned} AD = AB \sin B, \text{ and } AD &= AC \sin (180^\circ - C) \\ &= AC \sin C \text{ (Art. 95)}; \end{aligned}$$

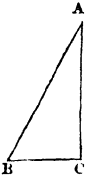
therefore $$AB \sin B = AC \sin C,$$

therefore $$\frac{AB}{AC} = \frac{\sin C}{\sin B},$$

that is $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be a *right* angle we have

$$AC = AB \sin B,$$



therefore $$\frac{AB}{AC} = \frac{1}{\sin B} = \frac{\sin 90^0}{\sin B} \text{ (Art. 99)},$$

that is $$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

Thus it is shewn that in every case

$$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

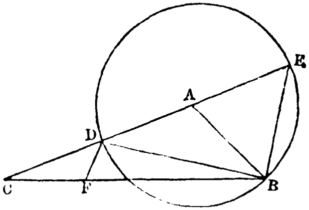
Similarly $$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{and} \quad \frac{a}{c} = \frac{\sin A}{\sin C}.$$

The results may be written symmetrically thus:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

and we shall shew hereafter that each of these is equal to $\frac{1}{2R}$, where R is the radius of the circle described round the triangle.

104. *In any triangle* ABC *the tangent of half the difference of the angles* B *and* C *is to the tangent of half their sum as the difference of the two sides* AC *and* AB *is to their sum.*



Let AB be the shorter of the two sides AB, AC. With centre A, and radius AB, describe a circle cutting AC at D. Produce CA to meet the circumference again at E. Join BD. Draw DF at right angles to BD, meeting BC at F.

The angle BAE is equal to the sum of the angles ABC and ACB, by Euclid I. 32; the angle BDE is equal to half the angle BAE, by Euclid III. 20;

thus $$BDE = \frac{1}{2}(B + C).$$

The angle ADB is equal to the sum of the angles DBC and DCB, by Euclid I. 32:

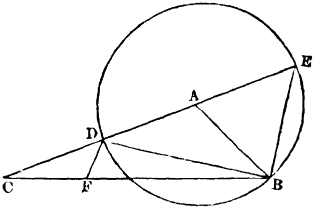
thus $$DBC = ADB - DCB = \frac{1}{2}(B + C) - C = \frac{1}{2}(B - C).$$

The angle DBE is a right angle, by Euclid III. 31:

thus $$\frac{BE}{BD} = \tan BDE.$$

And $$\frac{DF}{DB} = \tan DBC.$$

Therefore $$\frac{\tan\frac{1}{2}(B - C)}{\tan\frac{1}{2}(B + C)} = \frac{DF}{DB} \div \frac{BE}{DB} = \frac{DF}{BE}.$$



But DF is parallel to BE, by Euclid I. 27; therefore, by Euclid VI. 4,

$$\frac{DF}{DC}=\frac{BE}{CE},$$

therefore $$\frac{DF}{BE}=\frac{CD}{CE}=\frac{CA-AD}{CA+AE}=\frac{CA-AB}{CA+AB};$$

thus $$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{b-c}{b+c}.$$

105. The investigation of the preceding Article has been usually given in elementary works on Trigonometry; another method has been indicated in which two important formulæ are derived from the figure, and then the result just obtained is deduced.

We have shewn that the angle $ADB=\frac{1}{2}(B+C)$, and that the angle $DBC=\frac{1}{2}(B-C)$; thus $CBE=90^\circ+\frac{1}{2}(B-C)$. And the angle $CEB=\frac{1}{2}A$, by Euclid III. 20.

Now from the triangle CBE we have

$$\frac{CE}{CB}=\frac{\sin CBE}{\sin CEB}=\frac{\sin(180^\circ-CBE)}{\sin\frac{1}{2}A}=\frac{\sin\{90^\circ-\frac{1}{2}(B-C)\}}{\sin\frac{1}{2}A};$$

that is $$\frac{b+c}{a}=\frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A} \text{ (Art. 16)};$$

therefore $$(b+c)\sin\tfrac{1}{2}A=a\cos\tfrac{1}{2}(B-C)\ldots\ldots\ldots\ldots(1).$$

Again, from the triangle DCB we have

$$\frac{CD}{CB}=\frac{\sin CBD}{\sin CDB}=\frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)},$$

that is $$\frac{b-c}{a}=\frac{\sin \frac{1}{2}(B-C)}{\sin (90^0-\frac{1}{2}A)}=\frac{\sin \frac{1}{2}(B-C)}{\cos \frac{1}{2}A} \text{ (Art. 16)};$$

therefore $$(b-c)\cos \tfrac{1}{2}A=a\sin \tfrac{1}{2}(B-C)\text{...............(2).}$$

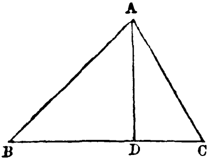
Divide (2) by (1); then

$$\tan \tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot \tfrac{1}{2}A=\frac{b-c}{b+c}\cot \frac{180^0-B-C}{2}$$

$$=\frac{b-c}{b+c}\cot\left(90^0-\frac{B+C}{2}\right)=\frac{b-c}{b+c}\tan \tfrac{1}{2}(B+C) \text{ (Art. 16)};$$

this agrees with the result already obtained.

106. *To express the cosine of an angle of a triangle in terms of the sides.*



Let ABC be a triangle, and suppose C an acute angle. Draw AD perpendicular to BC.

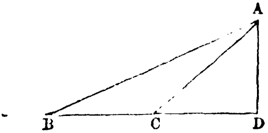
Then, by Euclid II. 13,

$$AB^2=BC^2+AC^2-2BC\,.\,CD,$$

and $$CD=AC\cos C;$$

therefore $$c^2=a^2+b^2-2ab\cos C.$$

Next suppose C an obtuse angle. Draw AD perpendicular to BC produced.



Then, by Euclid II. 12,

$$AB^2 = BC^2 + AC^2 + 2BC \,.\, CD;$$

and $CD = AC \cos ACD = AC \cos(180^\circ - C) = -AC \cos C$ by Art. 95;

therefore $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Thus in both cases

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Moreover when C is a right angle $a^2 + b^2 = c^2$, and $\cos C = 0$, by Art. 99. Thus the formula just found for $\cos C$ is true whatever the angle C may be.

Similarly

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}.$$

107. *To express the sine, the cosine, and the tangent, of half an angle of a triangle in terms of the sides.*

We have by the preceding Article

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

therefore $$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}.$$

Hence, by Art. 34,

$$(\sin \tfrac{1}{2} A)^2 = \frac{a^2 - (b-c)^2}{4bc}$$

$$= \frac{(a+b-c)(a+c-b)}{4bc}.$$

Let $2s$ stand for $a+b+c$, so that s is half the sum of the sides of the triangle; then

$$a+b-c = a+b+c-2c = 2s-2c = 2(s-c),$$
$$a+c-b = a+b+c-2b = 2s-2b = 2(s-b).$$

Therefore $$(\sin \tfrac{1}{2} A)^2 = \frac{(s-b)(s-c)}{bc},$$

and $$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Also $$1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}.$$

Hence, by Art. 34,

$$(\cos \tfrac{1}{2} A)^2 = \frac{(b+c)^2 - a^2}{4bc}$$

$$= \frac{(a+b+c)(b+c-a)}{4bc} = \frac{s(s-a)}{bc};$$

and $$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$$

From the values of $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$ we deduce

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similar expressions hold for the sine, the cosine, and the tangent of half of each of the other angles.

108. *To express the sine of an angle of a triangle in terms of the sides.*

By Art. 34, $\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$,

$$\text{therefore } \sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Or we may proceed thus: $(\sin A)^2 = 1 - (\cos A)^2$

$$= 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2 = \frac{4b^2c^2 - (b^2+c^2)^2 - a^4 + 2a^2(b^2+c^2)}{4b^2c^2}$$

$$= \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{4b^2c^2};$$

$$\text{therefore} \qquad \sin A = \frac{\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}}{2bc}.$$

Similar expressions hold for $\sin B$ and $\sin C$.

By comparing the two expressions for $\sin A$ we infer that

$$s(s-a)(s-b)(s-c) = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{16},$$

and this can be verified by multiplying out the factors s, $s-a$, $s-b$, and $s-c$. For

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)$$

$$= \{(b+c)^2 - a^2\}\{a^2 - (c-b)^2\}$$

$$= (2bc + b^2 + c^2 - a^2)(2bc + a^2 - b^2 - c^2)$$

$$= 4b^2c^2 - (b^2+c^2-a^2)^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

EXAMPLES. X.

1. If $\sin B = \frac{1}{4}$, $a = 3$, $b = \frac{3}{2}$, find A.

2. If $A = 75^\circ$, $B = 45^\circ$, $b = 2$, shew that $a = \sqrt{3} + 1$.

3. If $c = b(\sqrt{3} - 1)$, and $A = 30^\circ$, find B and C.

4. If $b = 2a$, and $C = 60^\circ$, find A, B, and c.

5. Find A when $a = 7$, $b = 5$, $c = 3$.

6. If a, b, and c are $1\frac{1}{5}$ feet, $1\frac{3}{5}$ feet, and 2 feet respectively, find C.

7. The sides of a triangle are 7, 8, 13: find the greatest angle.

8. The sides of a triangle are respectively 13 and 15 feet, and the cosine of the included angle is $\frac{33}{65}$: find the third side, and also the perpendicular on it from the given angle.

9. Shew from the formulæ for $\sin B$ and $\sin \frac{C}{2}$ that $B = \frac{C}{2}$ if $c^2 = b(b + a)$.

10. If $a = 5$, $b = 4$, and $C = 60^\circ$, find c; having given

$$\log 3 = \cdot 4771213, \qquad \log 7 = \cdot 8450980.$$

$$\log 45825 = 4\cdot 6611025, \qquad \log 45826 = 4\cdot 6611120.$$

11. A perpendicular is drawn from the angle A of a triangle on the side BC meeting it at D; and a perpendicular from B on the side CA meeting it at E: shew that if the angle C is acute, $DE = c \cos C$.

12. Shew immediately from the figures in Art. 106 that

$$a = b \cos C + c \cos B.$$

13. From the result in the preceding example and the two analogous results deduce the value of $\cos C$ which is given in Art. 106.

14. If $\sin A = p\sqrt{(1-q^2)} + q\sqrt{(1-p^2)}$, find $\cos A$.

15. Shew that $\tan A = \dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$.

16. If $\tan A + \cot A = 2$, then $\sin A + \cos A = \sqrt{2}$.

17. Find A from the equation

$$\sec^2 A + \operatorname{cosec}^2 A = 3 \sec^4 A.$$

18. The hypotenuse AB of a right-angled triangle is divided at D so that AD is to BD as CB is to CA: shew that

$$\tan ACD = \frac{a^2}{b^2}, \qquad CD = \frac{\sqrt{a^4 + b^4}}{a+b}.$$

19. From a ship at sea it is observed that the angle between two forts A and B is α; the ship sails for m miles towards A, and the angle between the forts is then observed to be β: find the distance of the ship from B at the second observation.

20. If

$$a \cos^2 x + b \sin^2 x = m \cos^2 y,$$

$$a \sin^2 x + b \cos^2 x = n \sin^2 y,$$

and

$$m \tan^2 x = n \tan^2 y,$$

shew that

$$(a+b)(m+n) = 2mn,$$

and

$$\tan^2 x = 1.$$

XI. *Solution of Triangles.*

109. *To solve a triangle having given two angles and a side.*

Suppose A and C the given angles, and b the given side; then

$$B = 180^0 - A - C;$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \qquad \text{(Art. 103)}$$

therefore $$a = \frac{b \sin A}{\sin B};$$

therefore $$\begin{aligned}\log a &= \log b + \log \sin A - \log \sin B \\ &= \log b + L \sin A - 10 - L \sin B + 10 \\ &= \log b + L \sin A - L \sin B.\end{aligned}$$

Similarly $\log c = \log b + L \sin C - L \sin B$.

110. *To solve a triangle having given two sides and the included angle.*

Suppose b and c the given sides and A the included angle.

We have, by Art. 104,

$$\frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)} = \frac{b - c}{b + c}.$$

Now $$A + B + C = 180^0,$$

therefore $$\tfrac{1}{2}(B + C) = 90^0 - \frac{A}{2},$$

therefore $$\tan \tfrac{1}{2}(B + C) = \cot \frac{A}{2}. \quad \text{(Art. 16.)}$$

Hence $$\frac{\tan \frac{1}{2}(B-C)}{\cot \frac{1}{2}A} = \frac{b-c}{b+c};$$

therefore $$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \tfrac{1}{2}A,$$

therefore

$$\log \tan \tfrac{1}{2}(B-C) = \log (b-c) + \log \cot \tfrac{1}{2}A - \log (b+c),$$

therefore

$$L \tan \tfrac{1}{2}(B-C) = \log (b-c) + L \cot \tfrac{1}{2}A - \log (b+c).$$

This formula determines $\frac{1}{2}(B-C)$; and $\frac{1}{2}(B+C)$ is known, since it is $90^0 - \frac{1}{2}A$; then B and C can be immediately found.

Also $$\frac{a}{c} = \frac{\sin A}{\sin C},$$

therefore $$\log a = \log c + L \sin A - L \sin C,$$

from which a can be found.

Or a may be found from one of the formulæ investigated in Art. 105. We have

$$a \cos \tfrac{1}{2}(B-C) = (b+c) \sin \tfrac{1}{2}A,$$

therefore $\log a = \log (b+c) + L \sin \frac{1}{2}A - L \cos \frac{1}{2}(B-C)$.

In this way of finding a we only require *two* additional logarithms, as $\log (b+c)$ is already known; whereas in the former way we required *three* additional logarithms: moreover $L \sin \frac{1}{2}A$ can be taken out of the tables at the same opening of them as $L \cot \frac{1}{2}A$, thus saving trouble.

These processes suppose b and c to be unequal. If b and c are equal, the triangle is isosceles; and then by drawing a perpendicular from the angle A to the base the triangle may be divided into two right-angled triangles, and the solution may be completed as in Art. 78. Or, as all the angles are known, we may find the other side by Art. 109.

111. *To solve a triangle having given two sides and the angle opposite to one of them.*

Let a and b be the given sides, and A the given angle; then

$$\frac{\sin B}{\sin A} = \frac{b}{a},$$

therefore $$\sin B = \frac{b}{a} \sin A,$$

therefore $$\log \sin B = \log b + \log \sin A - \log a,$$

therefore $$L \sin B = \log b + L \sin A - \log a.$$

If $\frac{b \sin A}{a}$ is less than unity, two different angles may be found less than 180^0 which have $\frac{b \sin A}{a}$ for sine, one of these angles being less than a right angle, and the other greater. If a be not less than b, then A must be not less than B, and therefore B must be an acute angle; thus only the smaller value is admissible for B. If a be less than b, then either value may be taken for B. When B is determined, C is known, since it is $180^0 - A - B$, and then c can be found from

$$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

Thus if two values are admissible for B we obtain two corresponding values for C and c, so that two triangles can be found from the given elements.

If $\frac{b \sin A}{a} = 1$, then B is a right angle, so that only one triangle can be found from the given elements.

If $\frac{b \sin A}{a}$ is greater than unity, no triangle exists with the given elements.

Thus when two sides are given and the angle opposite to the less, we can *generally* find two triangles from the given elements, and this case in the solution of triangles is therefore called the *ambiguous* case.

We say that two triangles can be *generally* found, in order to allow for the exceptions: for the triangle may be *right-angled*, and then only one triangle can be found; or the triangle may be impossible.

Some figures illustrating the solution of triangles when the given elements are two sides and an opposite angle will be found in Art. 88.

112. *To solve a triangle having given the three sides.*

By Art. 107, we have

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

and similar formulæ are true for the other half angles. Any one of the three formulæ will serve for finding the angle $\frac{A}{2}$; the formulæ for the tangent will however be the best to use with logarithms, because then we only require the logarithms of s, $s-a$, $s-b$, and $s-c$ in order to find *all* the angles; whereas if we use the formulæ for the sine or cosine, we shall require also the logarithms of the sides.

In practice it is often found convenient when all the angles are required to calculate them from the formulæ

$$\tan\frac{A}{2} = \frac{S}{s(s-a)}, \quad \tan\frac{B}{2} = \frac{S}{s(s-b)}, \quad \tan\frac{C}{2} = \frac{S}{s(s-c)},$$

where S denotes $\sqrt{s(s-a)(s-b)(s-c)}$. Then we obtain a *verification* of the work from the condition that $A+B+C$ must be equal to 180°.

113. We will now take some examples of the solution of triangles; we shall not give the *process* of applying the principle of proportional parts, for this has been sufficiently exemplified in Chapters VI. and VII.: we shall merely state the results.

Example (1). Given $b=234{\cdot}7$, $c=185{\cdot}4$, $A=84^\circ\, 36'$.

$$L\tan\tfrac{1}{2}(B-C)=\log(b-c)+L\cot\tfrac{1}{2}A-\log(b+c).$$

$$b-c=49{\cdot}3, \qquad b+c=420{\cdot}1, \qquad \tfrac{1}{2}A=42^\circ\, 18';$$

$$\begin{array}{rl}
\log 49{\cdot}3 = & 1{\cdot}6928469 \\
L\cot 42^\circ\, 18' = & 10{\cdot}0409920 \\
\hline
 & 11{\cdot}7338389 \\
\log 420{\cdot}1 = & 2{\cdot}6233527 \\
\hline
L\tan\tfrac{1}{2}(B-C) = & 9{\cdot}1104862
\end{array}$$

Hence we find from the tables

$$\tfrac{1}{2}(B-C)=7^\circ\, 20'\, 56''\ \text{nearly},$$

and $$\tfrac{1}{2}(B+C)=47^\circ\, 42'.$$

Thus $B=55^\circ\, 2'\, 56''$, $C=40^\circ\, 21'\, 4''$.

Then $$\log a=\log c+L\sin A-L\sin C$$

$$\begin{array}{rl}
\log 185{\cdot}4 = & 2{\cdot}2681097 \\
L\sin 84^\circ\, 36' = & 9{\cdot}9980683 \\
\hline
 & 12{\cdot}2661780 \\
L\sin 40^\circ\, 21'\, 4'' = & 9{\cdot}8112196 \\
\hline
\log a = & 2{\cdot}4549584
\end{array}$$

Hence we find $a=285{\cdot}0745$.

Or we may find a by the other method explained in Art. 110.

$$\log a = \log (b+c) + L \sin \tfrac{1}{2} A - L \cos \tfrac{1}{2} (B-C)$$

$$\begin{aligned} \log 420{\cdot}1 &= 2{\cdot}6233527 \\ L \sin 42^\circ\, 18' &= \underline{9{\cdot}8280231} \\ &\ 12{\cdot}4513758 \\ L \cos 7^\circ\, 20'\, 56'' &= \underline{9{\cdot}9964178} \\ \log a &= 2{\cdot}4549580 \end{aligned}$$

Hence we can find a as before.

The two modes of calculating $\log a$ give results which differ slightly; but $\frac{1}{2}(B-C)$ is in fact rather less than $7^\circ 20' 56''$, being about $7^\circ 20' 55''{\cdot}8$: if this more correct value be employed we shall find that both modes of calculating will agree in giving a result almost identical with that just obtained.

Example (2). Given $a = 283{\cdot}4$, $b = 348{\cdot}5$, $A = 32^\circ 15'$.

$$L \sin B = \log b + L \sin A - \log a,$$

$$\begin{aligned} \log 348{\cdot}5 &= 2{\cdot}5422028 \\ L \sin 32^\circ\, 15' &= \underline{9{\cdot}7272276} \\ &\ 12{\cdot}2694304 \\ \log 283{\cdot}4 &= 2{\cdot}4523998 \\ L \sin B &= 9{\cdot}8170306 \end{aligned}$$

$$B = 41^\circ\, 0'\, 36'', \text{ or } 138^\circ\, 59'\, 24''.$$

This is an example of the *ambiguous case*, so that there are two triangles with the given elements.

If we take $B = 41^\circ 0' 36''$ we have

$$C = 106^\circ\, 44'\, 24''.$$

Then $\log c = \log a + L \sin C - L \sin A$,

$$\begin{aligned} \log 283{\cdot}4 &= 2{\cdot}4523998 \\ L \sin 106^\circ\, 44'\, 24'' = L \sin 73^\circ\, 15'\, 36'' &= \underline{9{\cdot}9811940} \\ &\ 12{\cdot}4335938 \\ L \sin 32^\circ\, 15' &= \underline{9{\cdot}7272276} \\ \log c &= 2{\cdot}7063662 \end{aligned}$$

Hence we find $c = 508{\cdot}588$.

If we take $B = 138^\circ\ 59'\ 24''$, we have

$$C = 8^\circ\ 45'\ 36''.$$

$$\begin{array}{rr} \log 283{\cdot}4 = & 2{\cdot}4523998 \\ L \sin 8^\circ\, 45'\, 36'' = & 9{\cdot}1826882 \\ & 11{\cdot}6350880 \\ L \sin 32^\circ\, 15' = & 9{\cdot}7272276 \\ \log c = & 1{\cdot}9078604 \end{array}$$

Hence we find $c = 80{\cdot}8836$.

Example (3). Given $a = 15$, $b = 16$, $c = 17$.

Here $$s = \frac{1}{2}(15 + 16 + 17) = 24,$$

$$s - a = 9, \quad s - b = 8, \quad s - c = 7.$$

$$\begin{array}{rr} \log 8 = & {\cdot}9030900 \\ \log 7 = & \underline{{\cdot}8450980} \\ & 1{\cdot}7481880 \end{array} \qquad \begin{array}{rr} \log 24 = & 1{\cdot}3802112 \\ \log\ \ 9 = & \underline{{\cdot}9542425} \\ & 2{\cdot}3344537 \\ & 1{\cdot}7481880 \\ & 2){\cdot}5862657 \\ & {\cdot}2931328 \end{array}$$

$$L \tan \frac{A}{2} = 10 + \frac{1{\cdot}7481880}{2} - \frac{2{\cdot}3344537}{2}$$

$$= 10 - {\cdot}2931328 = 9{\cdot}7068672.$$

Hence we find $\dfrac{A}{2} = 26^\circ\ 59'\ 3''$.

Similarly we find $\dfrac{B}{2} = 29^\circ\ 48'\ 18''$.

Hence $$\frac{C}{2} = 90^\circ - \frac{A}{2} - \frac{B}{2} = 33^\circ\ 12'\ 39''.$$

EXAMPLES. XI.

1. Given $a = 274$, $A = 78^\circ$, $B = 54^\circ$: find b.

$\log 274 = 2{\cdot}4377506$, $L \sin 78^\circ = 9{\cdot}9904044$,
$\log 226{\cdot}62 = 2{\cdot}3552982$, $L \sin 54^\circ = 9{\cdot}9079576$.
$\log 226{\cdot}63 = 2{\cdot}3553174$,

2. Given $a = 254$, $B = 16^\circ$, $C = 64^\circ$: find b.

$\log 2{\cdot}54 = {\cdot}4048337$, $L \sin 80^\circ = 9{\cdot}9933515$,
$\log 7{\cdot}109 = {\cdot}8518085$, $L \sin 16^\circ = 9{\cdot}4403381$.
$\log 7{\cdot}11 = {\cdot}8518696$,

3. Given $a = 1000$, $B = 104^\circ$, $C = 24^\circ\, 29'\, 20''$: find b.

$\log 12396 = 4{\cdot}0932816$, $L \sin 76^\circ = 9{\cdot}9869041$,
$\log 12397 = 4{\cdot}0933166$, $L \sin 51^\circ\, 30' = 9{\cdot}8935444$,
$L \sin 51^\circ\, 31' = 9{\cdot}8936448$.

4. Given $b = 55$, $c = 45$, $A = 6^\circ$: find B and C.

$L \tan 87^\circ = 11{\cdot}2806042$, $L \tan 62^\circ\, 20' = 10{\cdot}2804451$,
$L \tan 62^\circ\, 21' = 10{\cdot}2807524$.

5. Given $b = 21$, $c = 10{\cdot}5$, $A = 36^\circ\, 52'\, 12''$: find B and C.

$\log 2 = {\cdot}3010300$, $L \cot 18^\circ\, 26'\, 6'' = 10{\cdot}4771213$.
$\log 15 = 1{\cdot}1760913$,

6. Given $b = 426$, $c = 354$, $A = 49^\circ\, 16'$: find B and C.

$\log 78 = 1{\cdot}8920946$, $L \cot 24^\circ\, 38' = 10{\cdot}3386231$,
$\log 72 = 1{\cdot}8573325$, $L \tan 11^\circ\, 22' = 9{\cdot}3032609$,
$L \tan 11^\circ\, 23' = 9{\cdot}3039143$.

7. Given $b = 100$, $c = 60$, $A = 42^\circ\, 30'$: find B and C.

$\log 2 = {\cdot}3010300$, $L \tan 32^\circ\, 44' = 9{\cdot}8080829$,
$L \cot 21^\circ\, 15' = 10{\cdot}4101858$, $L \tan 32^\circ\, 45' = 9{\cdot}8083606$.

8. Given $b = 82471$, $c = 63529$, $A = 43^{0}\ 10'$: find B and C.

log 18942 = 4·2774258, $L \tan 68^{0}\ 25' = 10·4027530$,
log 146 = 2·1643529, $L \tan 18^{0}\ 9'\ 20'' = 9·5157731$,
$L \tan 18^{0}\ 9'\ 30'' = 9·5158442$.

9. Given $b = 375400·1$, $c = 327762·9$, $A = 57^{0}\ 53'\ 16''·8$: find B and C.

log 703163 = 5·8470561, $L \cot \frac{A}{2} = 10·2572497$,
log 47637·2 = 4·6779462, $L \tan 6^{0}\ 59'\ 2''·4 = 9·0881398$.

10. Given $b = 5·75$, $c = 4·5$, $A = 48^{0}\ 20'$: find B, C, and a.

log 1·25 = ·0969100, $L \tan 65^{0}\ 50' = 10·3480258$,
log 4·3485 = ·6383382, $L \tan 15^{0}\ 12'\ 16'' = 9·4342119$,
log 4·5 = ·6532125, $L \sin 48^{0}\ 20' = 9·8733352$,
log 10·25 = 1·0107239, $L \sin 50^{0}\ 37'\ 44'' = 9·8882095$.

11. Given $a = 528$, $b = 252$, $A = 124^{0}\ 34'$: find B and C.

log 252 = 2·4014005, $L \sin 55^{0}\ 26' = 9·9156460$,
log 528 = 2·7226339, $L \sin 23^{0}\ 8' = 9·5942513$,
$L \sin 23^{0}\ 9' = 9·5945469$.

12. Given $a = 120$, $b = 80$, $A = 60^{0}$: find B, C and c.

log 2 = ·3010300, $L \sin 35^{0}\ 15' = 9·7612851$,
log 3 = ·4771213, $L \sin 35^{0}\ 16' = 9·7614638$,
log 1·3797 = ·1397847, $L \sin 84^{0}\ 44' = 9·9981626$,
log 1·3798 = ·1398161, $L \sin 84^{0}\ 45' = 9·9981743$.

13. Given $a = 21·217$, $b = 12·543$, $A = 29^{0}\ 51'$: find B and C.

log 21217 = 4·3266840, $L \sin 17^{0}\ 6'\ 40'' = 9·4686806$,
log 12543 = 4·0984014, $L \sin 17^{0}\ 6'\ 50'' = 9·4687490$,
$L \sin 29^{0}\ 51' = 9·6969947$.

14. Given $\log a = \cdot 9717973$, $\log b = \cdot 8929345$,
$A = 70^0\ 12'\ 10''$: find B, C, and $\log c$.
$L \sin 70^0\ 12'\ 10'' = 9\cdot 9735422$, $L \sin 58^0\ 6'\ 30'' = 9\cdot 9289325$.
$L \sin 51^0\ 41'\ 20'' = 9\cdot 8946794$,

15. Given $a = 36$, $b = 44$, $A = 32^0\ 42'$: find B and C.
$\log\ 3 = \cdot 4771213$, $L \sin 32^0\ 42' = 9\cdot 7325870$,
$\log 11 = 1\cdot 0413927$, $L \sin 41^0\ 19' = 9\cdot 8196888$,
$L \sin 41^0\ 20' = 9\cdot 8198325$.

16. Given $a = 10$, $b = 15$, $L \sin A = 9\cdot 5228787$,
$\log 3 = \cdot 4771213$: find B.

17. Given $a = 222$, $b = 318$, $c = 406$: find A.
$\log 473 = 2\cdot 6748611$, $\log 251 = 2\cdot 3996737$,
$\log 406 = 2\cdot 6085260$, $L \cos 16^0\ 28' = 9\cdot 9818117$,
$\log 318 = 2\cdot 5024271$, $L \cos 16^0\ 29' = 9\cdot 9817744$.

18. Given $a = 11$, $b = 13$, $c = 16$: find all the angles.
$\log 2 = \cdot 3010300$, $L \cot 21^0\ 31' = 10\cdot 4042321$,
$\log 3 = \cdot 4771213$, $L \cot 21^0\ 32' = 10\cdot 4038620$,
$\log 7 = \cdot 8450980$, $L \cot 41^0\ 35' = 10\cdot 0519190$,
$L \cot 41^0\ 36' = 10\cdot 0516645$.

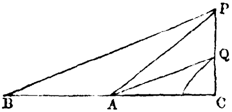
19. Given $a = 25$, $b = 26$, $c = 27$: find all the angles.
$\log 35 = 1\cdot 5440680$, $L \tan 28^0\ 7'\ 30'' = 9\cdot 7279568$,
$\log\ 3 = \cdot 4771213$, $L \tan 28^0\ 7'\ 40'' = 9\cdot 7280074$,
$L \tan 31^0\ 56'\ 50'' = 9\cdot 7948986$,
$L \tan 31^0\ 57' = 9\cdot 7949455$.

20. If the angles of a plane triangle be in Arithmetical Progression, and the greatest side be to the least as 5 is to 4, find all the angles.
$\log 3 = \cdot 4771213$, $L \tan 10^0\ 53'\ 30'' = 9\cdot 2842475$,
$L \tan 10^0\ 53'\ 40'' = 9\cdot 2843610$.

XII. *Heights and Distances.*

114. We have already in Arts. 38...45 considered some of the cases which properly belong to the present Chapter; and the student is advised to read those Articles again before proceeding to the additional investigations which we shall now give.

115. *To find the height of an inaccessible object placed on a hill.*



Let P be the top of the object, PQ its height; let A and B be two points in a horizontal plane, such that P, Q, A, B are all in the same vertical plane. Let PQ produced meet BA produced at C. At A observe the angles PAC and QAC; at B observe the angle PBC; and measure AB.

The angle $APB = PAC - PBC$, and is therefore known.

Then $$\frac{AP}{AB} = \frac{\sin PBA}{\sin APB};$$

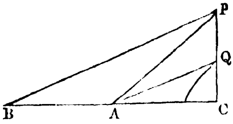
therefore $$AP = \frac{AB \sin PBA}{\sin APB}.$$

The angle $PAQ = PAC - QAC$, and is therefore known. The angle $PQA = QCA + QAC = 90^\circ + QAC$, and is therefore known.

Then $$\frac{PQ}{PA} = \frac{\sin PAQ}{\sin PQA};$$

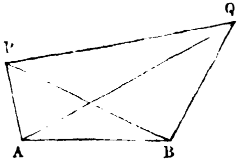
therefore $$PQ = \frac{PA \sin PAQ}{\sin PQA};$$

thus PQ is known.



Also CQ may be found, if required. For we may find AC from the triangle PAC, and then CQ from the triangle ACQ. Or we may find PC from the triangle PAC, and then by subtracting PQ we have CQ.

116. *To find the distance between two inaccessible points which are visible from two accessible points.*



Let P and Q be the inaccessible points; A and B the accessible points, from which P and Q are visible.

We will first suppose all the four points to be in the same plane.

At A observe the angles PAQ and QAB; at B observe the angles PBA and QBA; and measure AB.

Then in the triangle APB, the side AB and the angles PAB and PBA are known; thus PA can be found. Again, in the triangle ABQ, the side AB and the angles QAB and ABQ are known; thus AQ can be found. Lastly, in the triangle PAQ, the sides AP and AQ and the angle PAQ are known; thus PQ can be found.

If the four points are not all in the same plane the only difference is that we must observe the angle PAB as well as the angles PAQ and QAB; for now the angle PAB will not be equal to the sum of the angles PAQ and QAB.

117. Suppose that in the figure of the preceding Article we *know* PQ and wish to determine AB without measurement, having observed the same angles as before. We may adopt the following method :

We have
$$AP = \frac{AB \sin PBA}{\sin APB},$$
$$AQ = \frac{AB \sin QBA}{\sin AQB};$$

and, by Art. 106,
$$PQ^2 = AP^2 + AQ^2 - 2AP \,.\, AQ \cos PAQ:$$

therefore
$$PQ^2 = AB^2 \left\{ \frac{\sin^2 PBA}{\sin^2 APB} + \frac{\sin^2 QBA}{\sin^2 AQB} - \frac{2 \sin PBA \sin QBA \cos PAQ}{\sin APB \sin AQB} \right\}.$$

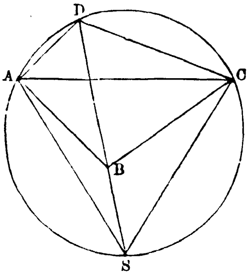
From this formula we can determine AB, since PQ and all the angles which occur are known ; but the formula is not suited for the application of logarithms. Hence the following method is practically better :

Suppose a second figure *similar* to $ABQP$ and denote it by $abqp$; then we shall have
$$AB \quad ab$$

Hence *assume* any value for ab, and calculate pq from the observed angles by the method of Art. 116 ; then find AB from the relation just given.

118. The problem in the next Article is of practical importance for military and other purposes. It may be necessary to know the distance of a station from some inaccessible objects ; by the aid of a good map the distances of three prominent objects from each other may be ascertained, and then by observing angles and by calculation we can determine the distance of a station from these objects. This we shall now shew.

119. *Given the distances of three points from each other, to find their distances from a station in the same plane with them.*

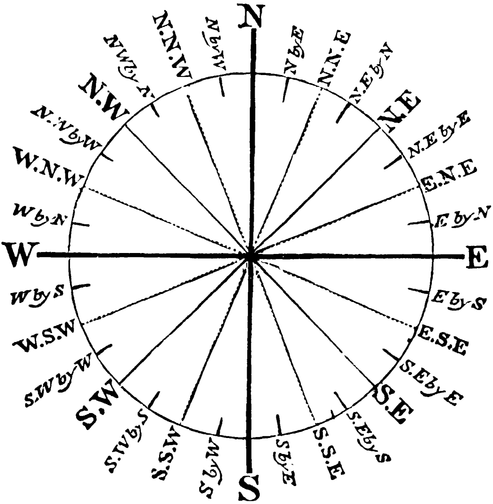


Let A, B, C denote the three points, so that AB, BC and CA are all known. Let the angles which AB and BC subtend at the station be observed. At the point A make the angle CAD equal to the observed angle at the station subtended by BC; and at the point C make the angle ACD equal to the observed angle at the station subtended by AB. Describe a circle round the triangle ACD, and produce DB to meet the circumference again at S. Then S will denote the station.

For the angle BSC = the angle CAD, and the angle BSA = the angle ACD; by Euclid III. 21. Hence the angles subtended at S by BA and BC are equal to the angles observed at the station; and therefore S denotes the station. Thus SA, SB and SC represent the three required distances: and we shall now shew how they may be calculated.

In the triangle ADC the side AC and the angles CAD and ACD are known; thus AD can be found. In the triangle ABC all the sides are known; thus the angle BAC can be found. Hence the angle BAD is known. In the triangle BAD the sides AB and AD, and the angle BAD are known; thus the other angles can be found. Hence the angle ABS is known. In the triangle ABS the side AB and the angles ASB and ABS are known; thus the sides SA and SB can be found. In the triangle ASC the sides AC and AS and the angle ASC are known; thus the side SC can be found.

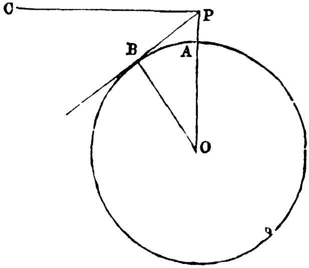
120. *Mariner's Compass.* In some problems the directions of straight lines are indicated by the terms used in the *Mariner's Compass*, which we will now explain.



The circumference of a circle is divided into 32 equal parts. Thus there are 32 *Points*, which have the names indicated in the figure, where N. S. E. and W. stand for North, South, East, and West respectively. The angle between two adjacent points is $\frac{360^\circ}{32}$, that is $11\frac{1}{4}^\circ$.

Hence we can readily determine the angle between straight lines drawn in directions which are indicated by these names. For example, if one object appear N. E. of an observer and another E.S.E. the angle between straight lines drawn from the observer towards these two objects is $6 \times 11\frac{1}{4}°$, that is $67\frac{1}{2}°$.

121. *Dip of the Horizon.* The earth is known to be very nearly spherical in shape. Suppose that from any point above the earth's surface straight lines are drawn to touch the earth. Then neglecting the inequalities of the earth's surface these straight lines will touch the earth at points which lie in the circumference of a circle. This circle bounds the portion of the earth's surface which is visible to a spectator at the point from which the straight lines are drawn, and is called the *terrestrial horizon* of the spectator.



Thus let O denote the centre of the earth, P a point above the surface, PB a straight line drawn from P to touch the surface at B: then B is a point on the horizon. Draw PC at right angles to PO in the same plane as B and O; then the angle CPB is called the *dip of the horizon* at P.

Let OP cut the surface of the earth at A. Let the angle BPC be denoted by θ. The following relations hold:

The angle $BOP = BPC = \theta$, so that $OP = OB \sec\theta$,

$$AP = OP - OA = OA(\sec\theta - 1) = \frac{OA(1-\cos\theta)}{\cos\theta},$$

$$PB = OB\tan\theta = \frac{AP\cos\theta}{1-\cos\theta}\tan\theta = \frac{AP\sin\theta}{1-\cos\theta}$$

$$= AP\cot\frac{\theta}{2}, \text{ by Art. 34.}$$

These relations are useful in solving problems relating to the dip of the horizon, and the distance of the horizon.

A formula which depends only on Geometry, may be conveniently noticed here. We know by Euclid III. 36 that

$$PB^2 = PA\,(PA + 2OA);$$

now in all cases which can occur in practice PA is extremely small compared with $2OA$, so that we have very approximately

$$PB^2 = 2PA \,.\, OA.$$

The number of miles in OA is very approximately 3960; let n denote the number of miles in PB: then the number of *feet* in PA

$$= \frac{(n \times 5280)^2}{2 \times 3960 \times 5280} = \frac{528n^2}{2 \times 396} = \frac{2}{3} n^2.$$

Thus if $n = 1$ we have $h = \frac{2}{3}$ of a foot $= 8$ inches very nearly; if $n = 2$ we have $h = \frac{2}{3}$ of 4 feet $= 32$ inches very nearly.

The surface of still water though apparently plane is really curved; and it appears from the above calculation that an object less than 8 inches above the surface of still water will be invisible to an eye on the surface at the distance of a mile.

122. The term *angle of depression* sometimes occurs in problems of heights and distances: in such cases the spectator is supposed to be at a point P above the surface of the earth, and the angle of depression of any point is the angle between the straight line PC and the straight line drawn from P to the point, which is somewhere *below* PC. Similarly to a spectator at P the *angle of elevation* of a point is the angle between PC and the straight line drawn from P to the point which is somewhere *above* PC.

EXAMPLES. XII.

1. The angles of depression of the top and bottom of a column observed from a tower 108 feet high are 30^0 and 60^0 respectively: find the height of the column.

2. At the foot of a mountain the elevation of its summit is found to be 45^0. After ascending for one mile, at a slope of 15^0, towards the summit, its elevation is found to be 60^0. Determine the height of the mountain.

3. A and B are two stations on a hill side; the inclination of the hill to the horizon is 30^0; the distance between A and B is 500 yards. C is the summit of another hill in the same vertical plane as A and B, on a level with A, but at B its elevation above the horizon is 15^0. Find the distance between A and C.

4. A ship which is known to be sailing due East at 12 miles an hour, was observed at noon to be 15^0 to the East of South; at 1 h. 30 m. after noon the ship was seen in the South East: determine the distance of the ship at noon.

5. A person wishing to know the distance of a point C measures a straight line AB, and finds it to be 100 yards; he observes that the angles BAC and ABC are respectively $53^0\,20'$ and $59^0\,30'$: determine the distance of C from A.

$L \sin 59^0\,30' = 9{\cdot}9353204$, $\log 93489 = 4{\cdot}9707605$.
$L \sin 67^0\,10' = 9{\cdot}9645602$,

6. A person standing on the bank of a river observes the elevation of the top of a tree on the opposite bank to be 51^0; and when he retires 30 feet from the river's bank he observes the elevation to be 46^0: determine the breadth of the river.

$L \sin 46^0 = 9{\cdot}8569341$, $\log 3 = {\cdot}4771213$,
$L \sin 39^0 = 9{\cdot}7988718$, $\log 1{\cdot}55823 = {\cdot}1926316$.
$L \sin 5^0 = 8{\cdot}9402960$,

7. From the top of a hill I observe two milestones on the level ground in a straight line before me, and I find their angles of depression to be respectively 5^0 and 15^0: determine the height of the hill.

$L \sin 5^0 = 8{\cdot}9402960$, $\log 12990 = 4{\cdot}1136092$,
$L \sin 10^0 = 9{\cdot}2396702$, $\log 12991 = 4{\cdot}1136426$.
$L \sin 15^0 = 9{\cdot}4129962$,

8. A tower is situated on the top of a hill whose angle of inclination to the horizon is 30^0; the angle subtended by the tower at the foot of the hill is found by an observer to be 15^0; and on ascending 485 feet up the hill the tower is found to subtend an angle of 30^0: determine the height of the tower and the distance of its base from the foot of the hill.

$\log 3 = {\cdot}4771213$, $\log 280{\cdot}015 = 2{\cdot}4471813$.
$\log 485 = 2{\cdot}6857417$,

9. *ABCD* is a rectangular piece of water the dimensions of which are required, but on account of the nature of the ground the only measures which can be taken are the angles that *BC* subtends at *A*, and at a point *P* which is 220 feet from *A* in *BA* produced, the former being 71^0 and the latter 55^0. Find the length and breadth of the rectangle.

$L \sin 16^0 = 9{\cdot}4403381$, $\log 2{\cdot}2 = {\cdot}3424227$,
$L \sin 55^0 = 9{\cdot}9133645$, $\log 2{\cdot}12858 = {\cdot}3280900$,
$L \sin 71^0 = 9{\cdot}9756701$, $\log 6{\cdot}18186 = {\cdot}7911193$.
$L \cos 71^0 = 9{\cdot}5126419$,

10. A ship of a blockading squadron lies 4 miles to the South of a harbour, and observes that a ship leaves the harbour in a direction E. 30^0 S. If the blockading ship sails 12 miles an hour, find in what direction she must go so as to cross the course of the other ship in three quarters of an hour.

$L \sin 22^0\,38' = 9{\cdot}5852716$, $\log 2 = {\cdot}3010300$,
$L \sin 22^0\,39' = 9{\cdot}5855745$, $\log 3 = {\cdot}4771213$.

11. In a survey it is found necessary to continue a straight line AB past an obstacle, which, from its height, hinders the view of the parts beyond. A straight line BD is therefore measured at right angles to AB, and from the point D straight lines DP, DQ are drawn which clear the obstacle; the angles BDP and BDQ are found to contain 41^0 and 68^0 respectively, the distance BD being 180 yards. Determine the lengths which must be set off along DP and DQ to ensure that PQ shall be in the prolongation of AB.

$L \cos 41^0 = 9{\cdot}8777799$, $\log 1{\cdot}8 = {\cdot}2552725$,

$L \sin 22^0 = 9{\cdot}5735754$, $\log 2{\cdot}38502 = {\cdot}3774920$,

$\log 4{\cdot}80504 = {\cdot}6816970$.

12. The courses of two ships are N. and E. and their rates of sailing are equal; the bearing of the former with respect to the latter was E.N.E., but after each had sailed four miles the bearing was N.N.W.: determine the distance between the ships at the time of the first observation.

13. While sailing S.W. two ships are seen at anchor, one N.N.W., and the other W.N.W. After sailing 5 miles these ships are seen N. and N.W. respectively. Determine their bearing and distance from each other.

14. A ship sailing out of harbour is watched by an observer from the shore; and at the instant she disappears below the horizon he ascends to a height of 20 feet, and thus retains her in sight 40 minutes longer. Find the rate at which the ship is sailing, assuming the Earth to be a sphere of 4000 miles radius, and neglecting the height of the observer.

15. An observer from the deck of a ship 20 feet above the level of the sea can just see the top of a distant lighthouse, and on ascending to the masthead, which is 60 feet above the deck, he sees the door which he knows to be one-fourth of the height of the lighthouse above the level of the sea. Find his distance from the lighthouse and its height, assuming the earth to be a sphere of 4000 miles radius.

16. Two observers in the same horizontal plane stationed at a distance of 200 yards from each other observed the altitude and bearing of the top of a tower; to one of them the altitude was 60° bearing S.W., and to the other the altitude was 45° bearing W.: find the height of the tower.

17. A flag staff a feet high on the top of a tower is seen from a certain point in the horizontal plane on which the tower stands to subtend at the eye of the observer an equal angle (A) with the tower itself. Shew that the height of the tower $= a \cos 2A$.

18. A person walks p yards from A to E along AB the side of a triangle ABC, and observes the angle $AEC = \theta$; again he walks q yards from B to F along the side BA and observes the angle CFB also $= \theta$: the whole distance AB being c, shew how to solve the triangle ABC.

19. The angular elevations of the top of a tower are observed to be α, β, γ at the stations A, B, C respectively; the stations are in a horizontal straight line, the direction of which does not pass through the tower; the distance from A to B is p, and the distance from B to C is q, and B is between A and C: shew that $p \cot^2 \gamma + q \cot^2 \alpha$ is greater than $(p+q) \cot^2 \beta$.

20. The altitude of a cloud was observed to be α, and that of the sun in the same direction to be β, and the distance of the shadow of the cloud from the station of the observer was found to be c feet: shew that the height of the cloud was $\dfrac{c \sin \alpha \sin \beta}{\sin (\beta - \alpha)}$ feet.

21. A person walking along a straight road observes the greatest elevation of a tower to be β. From another straight road he observes the greatest elevation of the tower to be γ. The distances of the points of observation from the intersection of the two roads are b and c respectively. Shew that the height of the tower is

$$\sqrt{\frac{b^2 - c^2}{\cot^2 \gamma - \cot^2 \beta}}.$$

22. A balloon was observed to be due S. and at an elevation of 30^0 at noon; and its shadow was one mile from the place of observation, the Sun's altitude being 45^0. At 11 o'clock A.M., and 1 o'clock P.M., the bearings of the balloon were S.E. by S. and S.W. by W. respectively. Find the velocity and the direction of motion of the balloon, supposing it to move uniformly at a constant height. And shew that the nearest distance of the balloon from the place of observation was $\frac{\sqrt{3}+1}{4}\sqrt{10+3\sqrt{2}}$ miles.

In solving the following Examples the Tables will be required:

23. From a window it is observed that the angle of elevation of the top of a house on the opposite side of the street is 29^0, and the angle of depression of the bottom of the house is 56^0: determine the height of the house, supposing the breadth of the street to be 80 feet.

24. The elevations of two mountains in the same straight line with an observer are $9^0\,30'$ and $18^0\,20'$: on approaching four miles nearer they have both an elevation of 37^0. Find the heights of the mountains in yards.

25. P and Q are two inaccessible objects; a straight line AB, in the same plane as P and Q, is measured and found to be 280 yards; the angle PAB is 95^0, the angle QAB is $47^0\,30'$, the angle QBA is 110^0, and the angle PBA is $52^0\,20'$. Determine the length of PQ.

26. A, B, C are three objects at known distances apart; namely $AB = 1056$ yards, $AC = 924$ yards, and $CB = 1716$ yards. An observer places himself at a station S from which C appears directly in front of A, and observes the angle CSB to be $14^0\,24'$. Find the distance CS.

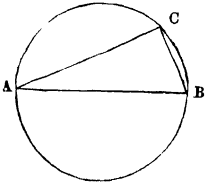
27. A, B, C are three objects at known distances apart; namely $AB = 320$ yards, $AC = 600$ yards, $BC = 435$ yards. From a station S it is observed that $ASB = 15^0$, and $BSC = 30^0$. Find the distances of S from A, B, and C; the point B being nearest to S, and the angle ASC being the sum of the angles ASB and BSC.

XIII. *Geometrical Solutions.*

123. The present Chapter will consist of Geometrical Solutions of some Trigonometrical Problems.

124. *To construct an angle with a given sine or cosine.*

Suppose we require an angle the *sine* of which is a given quantity *a*.



Describe a circle with unity for its diameter, and draw any diameter AB of this circle. With centre B, and radius equal to a, describe a circle; let C be one of the points where the circumferences of the two circles intersect: join AC and BC.

Then ACB is a right angle, by Euclid III. 31; and therefore the sine of BAC is $\frac{BC}{AB}$, that is a. Therefore BAC is such an angle as is required.

If we require an angle the *cosine* of which is a given quantity a, then the same construction may be made; and ABC will be such an angle as is required.

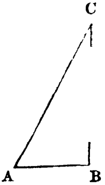
125. If an angle is required to have a given *cosecant*, then since the cosecant is the reciprocal of the sine the angle must have a known sine, and therefore may be found by the preceding Article.

Similarly, if an angle is required to have a given *secant* or a given *versed sine*, the angle must have a known cosine, and therefore may be found by the preceding Article.

126. *To construct an angle with a given tangent or cotangent.*

Suppose we require an angle the tangent of which is a given quantity a.

If the tangent of an angle is a the cosine is $\frac{1}{\sqrt{(a^2+1)}}$, and the sine is $\frac{a}{\sqrt{(a^2+1)}}$. Therefore since these quantities are known, we can construct the angle by Art. 124. Or we may proceed independently thus:



Take a straight line AB the length of which is unity; draw BC at right angles to AB and equal to a, and join CA.

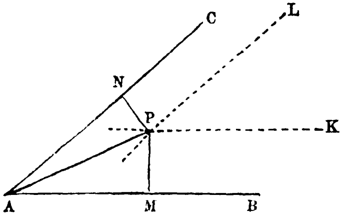
Then the tangent of BAC is $\frac{BC}{AB}$, that is a. Therefore BAC is such an angle as is required.

If we require an angle the *cotangent* of which is a given quantity a, then the same construction may be made; and ACB will be such an angle as is required.

127. *To divide a given angle into two parts which shall have their sines in a given ratio.*

Let BAC be the given angle, and let it be required to divide it into two parts, such that the sine of one part may be to the sine of the other part as m is to n.

Draw a straight line KP parallel to AB, and at a distance m from it; draw a straight line LP parallel to AC, and at a distance n from it. Let P be the point of intersection of these straight lines; join AP. Then AP shall divide the angle BAC in the required manner.



For draw PM perpendicular to AB, and PN perpendicular to AC. Then

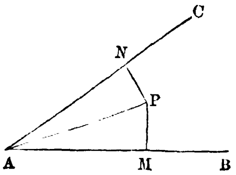
$$\sin PAB = \frac{PM}{AP} = \frac{m}{AP};$$

$$\sin PAC = \frac{PN}{AP} = \frac{n}{AP};$$

therefore

$$\frac{\sin PAB}{\sin PAC} = \frac{m}{AP} \div \frac{n}{AP} = \frac{m}{AP} \times \frac{AP}{n} = \frac{m}{n}.$$

128. *To divide a given angle into two parts which shall have their cosines in a given ratio.*



Let BAC be the given angle, and let it be required to divide it into two parts, such that the cosine of one part may be to the cosine of the other part as m is to n.

On AB take $AM=m$, and on AC take $AN=n$; draw MP at right angles to AB, and NP at right angles to AC. Let P be the point of intersection of these straight lines; join AP. Then AP shall divide the angle BAC in the required manner.

For $$\cos PAB=\frac{AM}{AP}=\frac{m}{AP};$$

$$\cos PAC=\frac{AN}{AP}=\frac{n}{AP}:$$

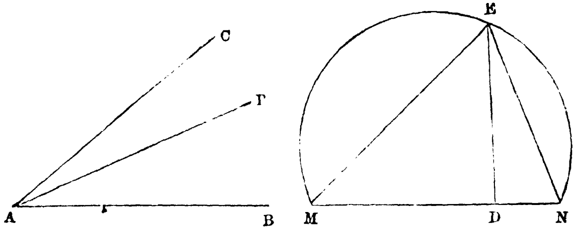
therefore

$$\frac{\cos PAB}{\cos PAC}=\frac{m}{AP}\div\frac{n}{AP}=\frac{m}{AP}\times\frac{AP}{n}=\frac{m}{n}.$$

If the point P does not fall *within* the angle BAC, we conclude that the angle cannot be divided into two parts in the proposed manner.

129. *To divide a given angle into two parts which shall have their tangents in a given ratio.*

Let BAC be the given angle, and let it be required to divide it into two parts such that the tangent of one part may be to the tangent of the other part as m is to n.



Take any straight line MN, and divide it at D so that DM may be to DN as m is to n. On MN describe a segment of a circle containing an angle equal to the angle BAC; draw DE at right angles to MN meeting the circumference at E. Join EM and EN. Then ED shall divide the angle MEN in the required manner.

For $$\tan DEM = \frac{DM}{DE}; \qquad \tan DEN = \frac{DN}{DE};$$

therefore $$\frac{\tan DEM}{\tan DEN} = \frac{DM}{DE} \div \frac{DN}{DE} = \frac{DM}{DN} = \frac{m}{n}.$$

At the point A, in the straight line AB, make the angle PAB equal to the angle DEN; then PAC is equal to DEM: and AP divides the angle BAC in the manner required.

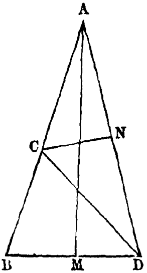
130. *To divide a given angle into two parts which shall have their cotangents in a given ratio.*

This may be solved as in the last Article. For with the notation there used we have

$$\cot DEM = \frac{DE}{DM}; \qquad \cot DEN = \frac{DE}{DN};$$

therefore $$\frac{\cot DEN}{\cot DEM} = \frac{DE}{DN} \div \frac{DE}{DM} = \frac{DM}{DN} = \frac{m}{n}.$$

131. By the aid of a problem in Euclid we can find an expression for the Trigonometrical Ratios of an angle of 18^0, as we will now shew.



In Euclid IV. 10 the following result is obtained:

ABD is a triangle in which $AB = AD$, and *C* is a point such that $AB \,.\, BC = AC^2$; and $AC = BD$: then each of the angles at *B* and *D* is double the angle at *A*.

Hence the angle at *A* is the fifth part of two right angles, and therefore contains 36^0; and each of the angles at *B* and *D* contains 72^0.

Draw *AM* perpendicular to *BD*; then the angle *BAM* contains 18^0.

Since $AB \,.\, BC = AC^2$, we have

$$AB(AB - AC) = AC^2;$$

thus

$$\frac{AC^2}{AB^2} + \frac{AC}{AB} - 1 = 0.$$

By solving the equation we obtain

$$\frac{AC}{AB} = \frac{-1 \pm \sqrt{5}}{2}$$

and we must take the upper sign, for $\frac{AC}{AB}$ is a positiv quantity. Thus

$$\frac{AC}{AB} = \frac{\sqrt{5} - 1}{2}.$$

Now $\sin 18^0 = \frac{BM}{AB} = \frac{1}{2}\frac{BD}{AB} = \frac{1}{2}\frac{AC}{AB} = \frac{\sqrt{5}-1}{4}$.

Hence we may deduce the values of the other Trigonometrical Ratios for an angle of 18^0. For

$$\cos^2 18^0 = 1 - \sin^2 18^0 = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \frac{6-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16};$$

therefore $$\cos 18^0 = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

And so on for the other Trigonometrical Ratios of 18^0.

Then, by Art. 16, we can find the Trigonometrical Ratios of 72^0.

Again, draw CN perpendicular to AD.

Then $$\cos 36^0 = \frac{AN}{AC};$$

and $AN = ND$, for the triangles ACN and DCN are equal in all respects, by the reasoning in Euclid IV. 10; thus

$$\cos 36^0 = \frac{1}{2}\frac{AD}{AC} = \frac{1}{\sqrt{5}-1} = \frac{(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{\sqrt{5}+1}{4}.$$

Or we may obtain this result without recurring again to the figure. For, by Art. 34,

$$\cos 36^0 = 1 - 2\sin^2 18^0 = 1 - \frac{2(\sqrt{5}-1)^2}{16} = 1 - \frac{3-\sqrt{5}}{4}$$

$$= \frac{\sqrt{5}+1}{4}.$$

Then $\sin 36^0 = \sqrt{(1-\cos^2 36^0)} = \frac{\sqrt{(10-2\sqrt{5})}}{4}$.

And so on for the other Trigonometrical Ratios of 36^0.

Then, by Art. 16, we can find the Trigonometrical Ratios of 54^0.

EXAMPLES. XIII.

1. A certain angle is equal to four times its complement: determine the angle.

2. The sine of A is equal to the cosine of B; and the number of degrees in A is three-fifths of the number of grades in B: determine the angles, supposing each of them less than a right angle.

3. Construct an angle whose tangent is four times its sine.

4. Divide a given angle into two parts, so that the sine of one part may bear a given ratio to the cosine of the other part.

5. Shew that $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.

6. Shew that the area of any quadrilateral figure is equal to half the product of the two diagonals into the sine of the angle between them.

7. In the *ambiguous* case when a, b and A are given, shew that the difference of the squares of the two values of the third side is $4b \cos A\sqrt{(a^2 - b^2 \sin^2 A)}$.

8. If $\dfrac{\sin \alpha}{\sin \beta} = \sqrt{2}$, and $\dfrac{\tan \alpha}{\tan \beta} = \sqrt{3}$, find α and β, supposing each of them less than a right angle.

9. If $\sin A \sin B = \sin \alpha \sin \beta$,

$$\cos A \cos B = \cos \alpha \cos \beta$$

and $$\cos^2 A + \cos^2 B - \cos^2 \gamma = 1;$$

then $$\sin^2 \alpha + \sin^2 \beta = \sin^2 \gamma.$$

10. If $\cos x = n \sin \alpha$, and $\cot x = \sin \alpha \cot \beta$, then

$$\cos^2 \beta = \frac{n^2}{1 + n^2 \cos^2 \alpha}.$$

11. If $\operatorname{cosec}^2\theta = m\tan\theta$, and $\sec^2\theta = n\cot\theta$,

then $$(mn)^3 = (\sqrt{m} + \sqrt{n})^4.$$

12. A rock is observed from the deck of a vessel to bear N.N.W.; and after the vessel has sailed 10 miles in the direction E.N.E. the same rock bears due W. Find the distance of the rock from the observer at the time of each observation.

13. AC is horizontal; AB, CD are two vertical towers. The angle of elevation of D observed at A is α, and observed at B is β; and $AB = h$. Find AC and CD.

14. In a triangle if $a = 10$, $b = 8$, $c = 12$, find the angles:

$$\log 2 = \cdot 3010300, \quad L\cos 41^\circ\, 24' = 9\cdot 8751256,$$

$$\log 3 = \cdot 4771213, \quad L\cos 41^\circ\, 25' = 9\cdot 8750142.$$

15. In a triangle if $a = 1$, $b = 7$, $c = \sqrt{(56)}$, find the angles:

$$\log 2 = \cdot 3010300, \quad L\cos 57^0\, 41'\, 10'' = 9\cdot 7279942,$$

$$\log 7 = \cdot 8450980, \quad L\cos 57^0\, 41'\, 20'' = 9\cdot 7279609.$$

16. A straight line CD subtends an angle α at a point A and also at a point B; and CA is at right angles to BD: shew that $AB = CD\cot\alpha$.

17. If $\cos\theta = \tan\lambda\cot\alpha$, $\cos\phi = \tan\lambda\cot\beta$, and

$$\sec\theta\sec\phi = \sec\lambda\tan\theta\tan\phi - \tan\alpha\tan\beta;$$

then $$\cos^2\lambda = \cos^2\alpha\cos^2\beta.$$

18. Find $\sin A$ and $\sin B$ from the equations

$$a\sin^2 A + b\sin^2 B = c, \quad a\sin 2A = b\sin 2B.$$

XIV. *Properties of Triangles.*

132. The present Chapter will contain some miscellaneous propositions chiefly relating to properties of triangles.

133. *To find an expression for the area of a triangle in terms of the sides.*

If b and c denote two sides of the triangle, and A the included angle, the area is $\frac{1}{2}bc \sin A$ by Art. 47. Now by Art. 108

$$\sin A = \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}};$$

therefore the area of the triangle $= \sqrt{\{s(s-a)(s-b)(s-c)\}}$.

By the same Article, we have the area of the triangle

$$= \tfrac{1}{4}\sqrt{(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)}.$$

It is usual to denote by S the expression

$$\sqrt{\{s(s-a)(s-b)(s-c)\}},$$

or
$$\tfrac{1}{4}\sqrt{(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)}.$$

134. *To find the radius of the circle inscribed in a triangle.*

Let ABC be a triangle; and let O be the centre of the circle inscribed in the triangle, and touching the sides at the points D, E, and F. Join OD, OE, and OF.

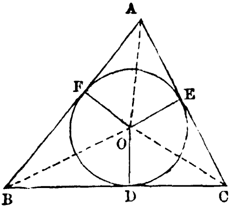
The angles at D, E, and F are right angles by Euclid III. 18.

Let r denote the radius of the circle. Then

$$\text{the area of the triangle } BOC = \tfrac{1}{2}BC \, . \, OD = \frac{ar}{2};$$

the area of the triangle $COA = \frac{1}{2} CA \, . \, OE = \frac{br}{2}$;

the area of the triangle $AOB = \frac{1}{2} AB \, . \, OF = \frac{cr}{2}$:



therefore, by addition,

$$\frac{a+b+c}{2} r = \text{the area of the triangle } ABC$$

$$= S, \text{ by Art. 133,}$$

that is, $sr = S$.

Therefore, $$r = \frac{S}{s}.$$

The radius of the inscribed circle is thus equal to the *area of the triangle divided by half the sum of the sides;* and various expressions can be obtained for the radius by employing the various expressions already given for the area of the triangle.

For example,

$$r = \frac{\sqrt{\{s(s-a)(s-b)(s-c)\}}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

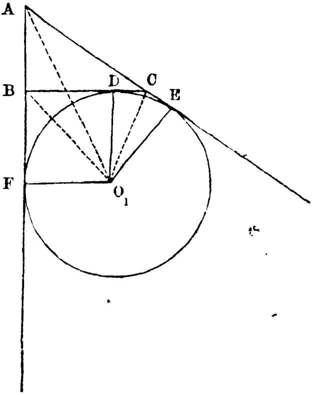
$$r = \frac{ab \sin C}{2s} = \frac{ab \sin C}{a+b+c}.$$

135. *To find the radius of the circle which touches one side of a triangle and the other two sides produced.*

Let ABC be a triangle; and let O_1 be the centre of the circle which touches BC and the other sides produced. Let D, E, and F be the points of contact. Join O_1D, O_1E, and O_1F.

The angles at D, E, and F are right angles by Euclid III. 18.

Let r_1 denote the radius of the circle.



The quadrilateral O_1BAC may be divided into the two triangles O_1AB, O_1AC; therefore the area of this quadrilateral is $\frac{c}{2}r_1 + \frac{b}{2}r_1$. Again the same quadrilateral may be divided into the two triangles O_1BC and ABC; therefore the area of this quadrilateral is $\frac{a}{2}r_1 + S$. Thus

$$\frac{c}{2}r_1 + \frac{b}{2}r_1 = \frac{a}{2}r_1 + S;$$

therefore $$\frac{c+b-a}{2} r_1 = S,$$

that is $$(s-a) r_1 = S.$$

Therefore $$r_1 = \frac{S}{s-a}.$$

Similarly let r_2 denote the radius of the circle which touches CA and the other sides produced; and let r_3 denote the radius of the circle which touches AB and the other sides produced: then we shall find that

$$r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c}.$$

A circle which touches one side of a triangle and the other sides produced is called an *escribed* circle.

136. Examples and problems are frequently given respecting the inscribed and escribed circles of a triangle, which depend chiefly on the following geometrical facts:

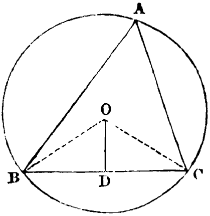
In the figure of Art. 134 the straight lines OA, OB, OC bisect the angles A, B, C respectively, by Euclid IV. 4. In the figure of Art. 135 the straight line O_1A bisects the angle A, the straight line O_1B bisects the angle FBD, and the straight line O_1C bisects the angle ECD. See *notes on the fourth book of Euclid.* Thus the points A, O, and O_1 are on one straight line, namely that which bisects the angle A.

And by geometry in both figures $AE = AF$, $BF = BD$, $CD = CE$.

Let O_2 denote the centre of the escribed circle which touches CA, then the points O_1, C, and O_2 are on one straight line, namely that which bisects the angle BCE, which is the supplement of ACB.

The angle OBO_1 is a right angle.

137. *To find the radius of the circle described roun a triangle.*



Let ABC be a triangle, and let O be the centre of th circle described round it. Draw OD perpendicular to BC then BC is bisected at D, by Euclid III. 3. Join OB an OC. Let R denote the radius of the circle.

The angle BOC is double of the angle BAC, by Eucli III. 20; therefore $BOD = A$.

And $$BD = R \sin A = \frac{a}{2};$$

therefore $$R = \frac{a}{2 \sin A}.$$

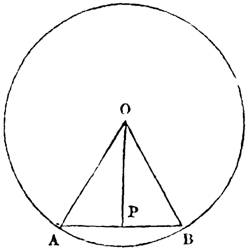
By Art. 108, $$\sin A = \frac{2S}{bc};$$

therefore $$R = \frac{abc}{4S}.$$

We see that $\frac{\sin A}{a} = \frac{1}{2R}$; thus we have, as stated i Art. 103,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}.$$

138. *To find the perimeter and the area of a regular polygon inscribed in a circle.*



Let O be the centre of a circle; let a regular polygon of n sides be inscribed in the circle, and let AB be one of the sides. Draw OP perpendicular to AB; and join OA, OB.

Let r denote the radius of the circle.

The angle $$AOB = \frac{360^0}{n},$$

$$AB = 2AP = 2AO \sin AOP = 2r \sin \frac{180^0}{n}.$$

The area of the triangle $AOB = \frac{1}{2} AB \,.\, OP = AP \,.\, OP$

$$= AO \sin AOP \,.\, AO \cos AOP = r^2 \sin \frac{180^0}{n} \cos \frac{180^0}{n}.$$

Or, by Art. 47, the area of the triangle

$$= \frac{1}{2} AO \,.\, BO \sin AOB = \frac{r^2}{2} \sin \frac{360^0}{n}.$$

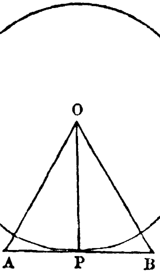
The perimeter of the regular polygon $= n \,.\, AB$

$$= 2nr \sin \frac{180^0}{n}.$$

The area of the regular polygon $= n \,.$ triangle AOB

$$= nr^2 \sin \frac{180^0}{n} \cos \frac{180^0}{n} = \frac{nr^2}{2} \sin \frac{360^0}{n}.$$

139. *To find the perimeter and the area of a regular polygon described about a circle.*



Let O be the centre of a circle; let a regular polygon of n sides be described about the circle, and let AB be one of the sides. Draw OP to the point of contact; then OP is perpendicular to AB, by Euclid III. 18. Join OA, OB.

Let R denote the radius of the circle.

The angle $$AOB = \frac{360^0}{n}.$$

$$AB = 2AP = 2OP \tan AOP = 2R \tan \frac{180^0}{n}.$$

The area of the triangle $AOB = \frac{1}{2} AB \,.\, OP = AP \,.\, OP$

$$= R^2 \tan \frac{180^0}{n}.$$

The perimeter of the regular polygon $= n \,.\, AB$

$$= 2nR \tan \frac{180^0}{n}.$$

The area of the regular polygon $= n$. triangle AOB

$$= nR^2 \tan \frac{180^0}{n}.$$

140. The expressions found in the two preceding Articles may be applied in various ways.

Thus in Art. 138 we have

$$AB = 2r \sin \frac{180^0}{n};$$

hence $$r = \frac{AB}{2} \operatorname{cosec} \frac{180^0}{n}:$$

this determines r when AB is given.

Similarly, from Art. 139 we have

$$R = \frac{AB}{2} \cot \frac{180^0}{n}:$$

this determines R when AB is given.

Again, suppose a regular polygon of n sides to be inscribed in a circle, and another regular polygon of n sides to be described about the *same* circle: then

$$\frac{\text{side of inscribed polygon}}{\text{side of circumscribed polygon}} = \frac{\sin \frac{180^0}{n}}{\tan \frac{180^0}{n}} = \cos \frac{180^0}{n};$$

$$\frac{\text{area of inscribed polygon}}{\text{area of circumscribed polygon}} = \frac{\sin \frac{180^0}{n} \cos \frac{180^0}{n}}{\tan \frac{180^0}{n}}$$

$$= \cos^2 \frac{180^0}{n}.$$

The second of these two ratios might also be deduced from the first by Euclid VI. 20.

EXAMPLES. XIV.

1. Shew from the figure in Art. 134 that

$$r(\cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C) = a.$$

2. Shew from the figure in Art. 135 that

$$r_1(\tan \tfrac{1}{2} B + \tan \tfrac{1}{2} C) = a.$$

3. Find the radius of the circle inscribed in an equilateral triangle.

4. Find the radius of an escribed circle when the triangle is equilateral.

5. Find the radius of the circle described about an equilateral triangle.

6. The sides of a triangle are 68, 75, and 77: determine the area, the radius of the inscribed circle, and the radius of the circumscribed circle.

7. In a triangle $a = 243$ yards, $b = 324$ yards, $c = 405$ yards: find the area.

8. Find the area of the triangle in which $a = 1864$, $b = c = 2796$.

$$\log 2 = \cdot 3010300, \qquad \log 24568 = 4\cdot 3903698,$$
$$\log 932 = 2\cdot 9694159, \qquad \log 24569 = 4\cdot 3903875.$$

9. Find the area of the triangle in which $a = 942$, $b = 812$, $c = 1270$.

$$\log 7 = \cdot 8450980, \qquad \log 1512 = 3\cdot 1795518,$$
$$\log 57 = 1\cdot 7558749, \qquad \log 3\cdot 82094 = \cdot 5821700,$$
$$\log 242 = 2\cdot 3838154.$$

10. Shew that in the figure of Art. 134

$$OA = \frac{c \sin \frac{1}{2} B}{\cos \frac{1}{2} C}.$$

11. Shew that in the figure of Art. 135

$$O_1A = \frac{c \cos \frac{1}{2} B}{\sin \frac{1}{2} C}, \qquad O_1B = \frac{a \cos \frac{1}{2} C}{\cos \frac{1}{2} A}.$$

12. Shew from the figure in Art. 134 that $AF = s - a$, and

$$r = (s - a) \tan \frac{A}{2}.$$

13. From the preceding result deduce the value of r in Art. 134.

14. Shew from the figure in Art. 135 that $AF = s$, and

$$r_1 = s \tan \frac{A}{2}.$$

15. From the preceding result deduce the value of r_1 in Art. 135.

16. Shew that

$$r_1 \cot \frac{A}{2} = r_2 \cot \frac{B}{2} = r_3 \cot \frac{C}{2}$$
$$= r \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right).$$

17. In the *ambiguous case,* when a, b and A are given shew that the circles circumscribing both triangles are equal in magnitude, and that the distance between their centres is

$$\sqrt{(a^2 \operatorname{cosec}^2 A - b^2)},$$

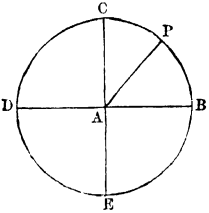
18. A tower is at right angles to a plane, and in the plane three points are found at which the tower subtends the same angle; these three points are distant from one another 123, 130, and 77 yards; also in passing from each one to the others in straight lines the greatest angle which the tower subtends at the eye is 45°. Shew that the height of the tower is $14\frac{5}{8}$ yards.

XV. *Angles greater than two right angles.*

141. In practical applications of Trigonometry it is not necessary to consider any angles except such as lie between zero and two right angles; and accordingly we have already explained the principles of the subject with sufficient generality for those who confine themselves to practical applications. But when Trigonometry is studied as a branch of theoretical mathematics it is found convenient to extend the notion of an angle; and this extension we shall now consider.

142. *Angles may be of any magnitude.*

Let BAD be any straight line, CAE a straight line at right angles to BAD. Suppose a straight line AP to revolve round one end A, starting from the position AB. When AP coincides with AC, the angle which has been described is a right angle; when AP coincides with AD, the angle which has been described is two right angles; when AP coincides with AE, the angle which has been described is three right angles; when AP coincides with AB, the angle which has been described is four right angles. Then as AP proceeds through a second revolution, the angle described will be greater than four right angles. Thus if AP be situated midway between AB and AC, the angle between AB and AP will be half a right

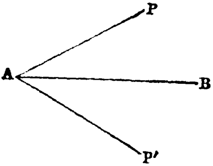


angle if AP be supposed in its *first* revolution; the angle will be four right angles and a half if AP be supposed in its *second* revolution; the angle will be eight right angles and a half if AP be supposed in its *third* revolution; and so on.

143. The straight lines CAE and BAD form by their intersection four right angles; these are called *quadrants: BAC* is called the *first quadrant, CAD* the *second quadrant, DAE* the *third quadrant*, and *EAB* the *fourth quadrant.* Now suppose any angle formed by the fixed straight line AB and the moveable straight line AP; if AP is situated in the first quadrant, the angle BAP is said to be in the first quadrant; if AP is situated in the second quadrant, the angle BAP is said to be in the second quadrant; and so on.

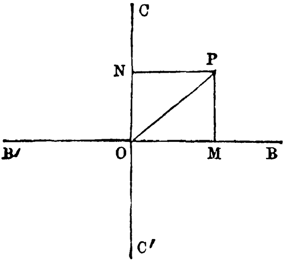
144. *Angles may be negative.*

By a convention similar to that in Art. 91, we distinguish angles measured in one direction from angles measured in the opposite direction. Let a straight line start from the position AB, and by revolving in one direction round A trace out the angle PAB, and let this angle be denoted by a *positive* number; then if the straight line start from AB, and by revolving round A in the opposite direction trace out the angle $P'AB$, this angle may be denoted by a *negative* number. If, for example, each of the angles BAP and BAP' be one-third of a right angle, and we denote the former by 30^0, the latter will be denoted by -30^0.



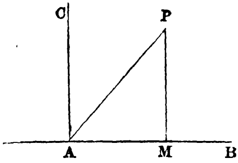
145. In Art. 91 we distinguish by means of positive and negative numbers between the two directions in which distances may be measured along a fixed straight line from a fixed point. In like manner we may distinguish between the two directions in which distances may be measured along a second straight line at right angles to the former.

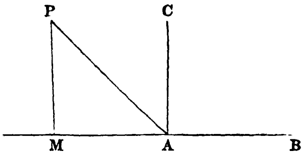
Let BOB', COC' be two straight lines which cut at right angles. Let P be any point in the plane which contains these two straight lines. The position of P will be known if we know the distance of P from each of the straight lines BB' and CC', and also know on *which side* of each of these straight lines P is situated. Draw PM and PN perpendicular to the straight lines BB' and CC' respectively. We shall adopt the following conventions: the distance NO or PM will be denoted by a *positive* number when P is *above* the straight line BB', and by a *negative* number when P is *below* the straight line BB'; the distance MO or PN will be denoted by a *positive* number when P is to the *right* of CC' and by a *negative* number when P is to the *left* of CC'.

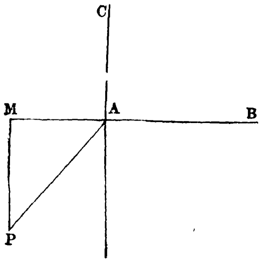


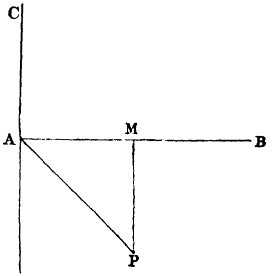
146. *General definitions of the Trigonometrical Ratios.*

We can now give general definitions of the Trigonometrical Ratios which will apply to angles of any magnitude.









Let AB, AC be two straight lines at right angles; let a straight line turn round the point A from AB towards AC and come into any position AP; draw PM perpendicular to AB or to AB produced through A. Then consider AP as always positive; consider AM as positive or negative according as M is on the same side of AC as B is, or on the opposite side; and consider PM as positive or negative according as P is on the same side of AB as C is, or on the opposite side. Let the angle PAB be denoted by A; then the Trigonometrical Ratios of A are thus defined:

$$\sin A = \frac{PM}{AP}, \quad \tan A = \frac{PM}{AM}, \quad \sec A = \frac{AP}{AM},$$

$$\cos A = \frac{AM}{AP}, \quad \cot A = \frac{AM}{PM}, \quad \operatorname{cosec} A = \frac{AP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \qquad \operatorname{covers} A = 1 - \sin A.$$

147. We have therefore Trigonometrical Ratios for any *positive* angle whatever may be its magnitude; and we have also Trigonometrical Ratios for any *negative* angle by adopting the convention that the Trigonometrical Ratios for any negative angle shall be the same as they would be for what we may call the *corresponding* positive angle. Thus, for example, in the last figure we may consider BAP as a negative angle, the magnitude of which is -60^0; then the Trigonometrical Ratios will be the same as for the angle formed by turning the moveable straight line AP in the positive direction until it reaches the position which it has in the figure; so that the Trigonometrical Ratios for the angle -60^0 will be the same as for the angle $360^0 - 60^0$.

148. It follows immediately from the definitions that if two angles differ by four right angles or by any multiple of four right angles, the Trigonometrical Ratios of the two angles are the same.

Thus for example any of the following angles has its sine equal to $\frac{1}{2}$, and its cosine equal to $\frac{\sqrt{3}}{2}$:

$$30^0,\ 390^0,\ 750^0,\ 1110^0, \ldots$$

149. The relations established in Arts. 19...23 between the Trigonometrical Ratios of angles not exceeding a right angle will now be seen to hold universally between the Trigonometrical Ratios of angles of any magnitude.

It will be sufficient for the student to satisfy himself that the following relations hold universally, as from these the others can be deduced:

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A},$$

$$\sec A = \frac{1}{\cos A}, \qquad \operatorname{cosec} A = \frac{1}{\sin A},$$

$$\sin^2 A + \cos^2 A = 1.$$

EXAMPLES. XV.

Find all the angles between 0 and 360^0 which satisfy the following twelve equations :

1. $\sin^2 A = \frac{1}{4}$.
2. $\cos^2 A = \frac{1}{2}$.
3. $\tan^2 A = 3$.
4. $\sin 2A = \frac{\sqrt{3}}{2}$.
5. $\cos 2A = -\frac{1}{2}$.
6. $\tan 2A = 1$.
7. $2\cos^2 A + \sin A = 1$.
8. $2\sin^2 A = 3(1 + \cos A)$.
9. $\tan^2 A - 4\tan A + 1 = 0$.
10. $\tan A - \cot A = 2$.
11. $\sin A = 1 - \cos 2A$.
12. $\cos A = 1 + \cos 2A$.

13. In any triangle, shew that

$$\frac{\operatorname{vers} A}{\operatorname{vers} B} = \frac{a(a+c-b)}{b(b+c-a)}.$$

14. In any triangle, shew that

$$\frac{\cot\frac{B}{2} + \cot\frac{C}{2}}{\cot\frac{A}{2}} = \frac{2a}{b+c-a}.$$

15. If a point be taken within a triangle so that the sides subtend equal angles at it, and α, β, γ be the distances of this point from the angular points of the triangle, shew that

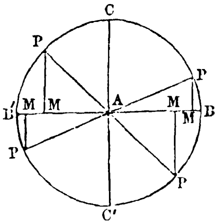
$$a^2 + b^2 + c^2 = 2(\alpha^2 + \beta^2 + \gamma^2) + \frac{4 \text{ area of the triangle}}{\sqrt{3}}.$$

16. If a triangle be divided into any two parts by a straight line drawn from one of the angles, shew that the radii of the circles described about these two triangles are in a ratio which is constant for all positions of the dividing straight line.

XVI. *Changes in the Ratios as the angle changes.*

150. In the present Chapter we shall trace the changes in magnitude and sign of the various Trigonometrical Ratios as the angle changes from zero to four right angles.

151. *To trace the changes in the sine of an angle as the angle varies.*



Let BAB' and CAC' be two straight lines at right angles, and suppose a straight line AP of constant length, to turn round one end A from the fixed position AB, so that P traces out the circle $BCB'C'$. From any position of P draw PM perpendicular to BAB'; then

$$\sin PAB = \frac{PM}{AP}.$$

When AP coincides with AB the perpendicular PM vanishes; thus when the angle is zero so also is its sine. While AP moves through the first quadrant PM is positive, and continually increases until AP coincides with AC, and then PM is equal to AP; thus as the angle increases from 0 to 90^0 the sine increases from 0 to 1. While AP moves through the second quadrant PM is positive and continually decreases until AP coincides with AB', and then PM vanishes; thus as the angle increases from 90^0 to 180^0 the sine diminishes from 1 to 0. While AP moves through the third quadrant PM is negative and increases *numerically* until AP coincides with AC'; thus as the angle increases from 180^0 to 270^0 the sine is *negative* and increases numerically from 0 to -1. While AP moves through the fourth quadrant PM is negative and decreases *numerically* until AP coincides with AB; thus as the angle increases from 270^0 to 360^0 the sine is *negative* and decreases numerically from -1 to 0.

152. *To trace the changes in the cosine of an angle as the angle varies.*

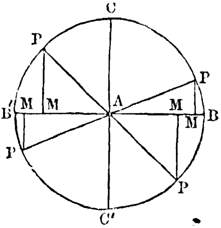
With the figure of Art. 151 we have $\cos PAB = \dfrac{AM}{AP}$. At first AP coincides with AB and then $AM = AP$, thus when the angle is zero the cosine is 1. While AP moves through the first quadrant AM is positive and continually decreases until AP coincides with AC and then AM vanishes; thus as the angle increases from 0 to 90^0 the cosine diminishes from 1 to 0. While AP moves through the second quadrant AM is negative and increases *numerically* until AP coincides with AB'; thus as the angle increases from 90^0 to 180^0 the cosine is negative and increases numerically from 0 to -1. While AP moves through the third quadrant AM is negative and decreases numerically until AP coincides with AC'; thus as the angle increases from 180^0 to 270^0 the cosine is *negative* and decreases numerically from -1 to 0. While AP moves through the fourth quadrant AM is positive and continually increases until AP coincides with AB; thus as the angle increases from 270^0 to 360^0 the cosine is positive and increases from 0 to 1.

153. *To trace the changes in the tangent of an angle as the angle varies.*

With the figure of Art. 151 we have $\tan PAB = \dfrac{PM}{AM}$.

At first AP coincides with AB, and then $AM = AB$; thus when the angle is zero so also is its tangent. While AP moves through the first quadrant PM and AM are positive; PM continually increases and AM continually decreases until AP coincides with AC; thus as the angle increases from 0 to 90^0 the tangent increases from 0 without limit, so that by taking an angle sufficiently near to 90^0 we can make the tangent as great as we please; this is usually expressed for the sake of abbreviation thus: *the tangent of* 90^0 *is infinite.* While AP moves through the second quadrant PM is positive and AM is negative;

PM continually decreases and *AM* increases *numerically* until *AP* coincides with *AB′*; thus as the angle increases from 90° to 180° the tangent is *negative* and decreases numerically from an indefinitely large value to zero. While *AP* moves through the third quadrant *PM* and *AM* are negative; *PM* increases *numerically* and *AM* decreases *numerically* until *AP* coincides with *AC′*; thus as the angle increases from 180° to 270°, the tangent is positive and increases from 0 without limit, so that by taking an angle sufficiently near to 270° we can make the tangent as great as we please; this as before is abbreviated thus: *the tangent of* 270° *is infinite.* While *AP* moves through the fourth quadrant *PM* is negative and *AM* is positive; *PM* continually decreases numerically, and *AM* increases until *AP* coincides with *AB*; thus as the angle increases from 270° to 360° the tangent is *negative* and decreases numerically from an indefinitely large value to zero.



Similarly the changes in the cotangent of an angle may be traced.

154. *To trace the changes in the secant of an angle as the angle varies.*

The changes in the secant of an angle may be traced by means of the figure in the same way as those of the sine, cosine, and tangent; or we may use the formula $\sec PAB = \frac{1}{\cos PAB}$, and infer the changes in the secant from the known changes in the cosine; we will adopt the latter method. As the angle increases from 0 to 90° the cosine diminishes from 1 to 0; thus the secant increases from 1 without limit, so we may say *the secant of* 90^0 *is infinite*. As the angle increases from 90^0 to 180^0 the cosine is *negative* and increases *numerically* from 0 to -1; thus the secant is *negative* and decreases *numerically* from an indefinitely large value to -1. As the angle increases from 180^0 to 270^0 the cosine is *negative* and decreases *numerically* from -1 to 0; thus the secant is *negative* and increases numerically from -1 to infinity. As the angle increases from 270^0 to 360^0 the cosine is positive and continually increases from 0 to 1; thus the secant is positive and diminishes from infinity to 1.

Similarly the changes in the cosecant of an angle may be traced.

155. *To trace the changes in the versed sine of an angle as the angle varies.*

Since $\text{vers}\,A = 1 - \cos A$, as the angle increases from 0 to 180^0 the versed sine increases from 0 to 2, and as the angle increases from 180^0 to 360^0 the versed sine diminishes from 2 to 0.

156. Thus we see that the sine and the cosine may have any value between -1 and $+1$; the tangent and the cotangent may have any value between $-\infty$ and $+\infty$; the secant and the cosecant may have any value between $-\infty$ and -1 and between $+1$ and $+\infty$. And it will be found on examination that no Trigonometrical Ratio changes its sign except when it passes through the value zero or the value infinity. The versed sine is always positive and may have any value between 0 and 2.

157. The student should carefully remember the signs of the Trigonometrical Ratios in the four quadrants: the following table exhibits them.

	1st	2nd	3rd	4th
sine	+	+	−	−
cosine	+	−	−	+
tangent	+	−	+	−

EXAMPLES. XVI.

Trace the changes in the sign and value of the following six expressions as A changes from 0 to 360^0:

1. $\sin A + \cos A$. 2. $\sin A - \cos A$.

3. $\sin^2 A$. 4. $\cos^2 A - \sin^2 A$.

5. $\sin A + \operatorname{cosec} A$. 6. $\tan A + \sec A$.

7. Find the least value of $\tan^2 A + \cot^2 A$ when A varies.

8. Find the least value of $4 \cos^2 A + \sec^2 A$ when A varies.

9. $ABCD$ is a quadrilateral figure which can be inscribed in a circle: shew that $AC \sin A = BD \sin B$.

10. Shew that the area of a regular hexagon inscribed in a circle is to the area of a regular octagon inscribed in the same circle as $3^{\frac{3}{2}}$ is to $2^{\frac{5}{2}}$.

11. The sides of a quadrilateral figure taken in order are 135, 180, 150 and 125 feet; and the angle contained by the first two sides is a right angle: determine the area of the figure.

12. ABC is a triangle, and D is the middle point of BC: shew that

$$AD^2 = \frac{1}{2}(b^2 + c^2) - \frac{a^2}{4}.$$

13. Shew that in any triangle the length of the perpendicular from A on the opposite side is equal to

$$\frac{b^2 \sin C + c^2 \sin B}{b + c}.$$

14. Shew that in any triangle

$$b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A.$$

15. Shew that the perpendicular from an angle of a triangle on the opposite side is an Harmonic Mean between the radii of the adjacent escribed circles.

16. If $A + B + C = 180^0$, shew that

$$\text{vers}\, A \,\text{vers}\,(B + C) = \sin A \sin (B + C).$$

17. ABC is a triangle; a straight line drawn from A meets BC at D, so that $BD = p$ and $CD = q$. Shew that

$$AD^2 = \frac{pb^2 + qc^2}{p+q} - pq.$$

With the notation of Arts. 134, 135, 137 establish the following results:

18. Perpendiculars are drawn from the angles of a triangle on the opposite sides meeting at K: shew that if the angle at A is acute

$$AK = 2R \cos A.$$

19. $OO_1 = (r_1 - r) \operatorname{cosec} \frac{1}{2} A = a \sec \frac{1}{2} A.$

20. $O_1O_2 = (r_1 + r_2) \sec \frac{1}{2} C = c \operatorname{cosec} \frac{1}{2} C.$

21. $$r = s \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}.$$

22. $$r = \frac{a - b}{\cot \frac{B}{2} - \cot \frac{A}{2}}.$$

23. $$(OO_1)^2 + (O_2O_3)^2 = \frac{a^2b^2c^2}{S^2}.$$

24. $$r \,.\, OO_1 = OB \,.\, OC.$$

25. $$OA \,.\, OO_1 = 4Rr.$$

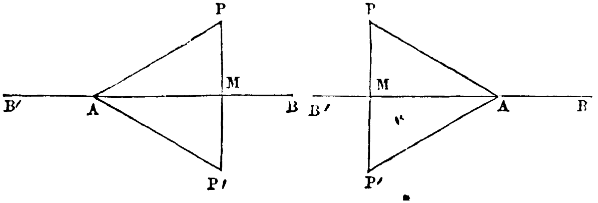
XVII. *Reduction of the angle.*

158. The object of the present Chapter is to shew that the Trigonometrical Ratios of any angle, positive or negative, can be expressed in terms of the Trigonometrical Ratios of an angle less than a right angle.

We shall require some preliminary propositions.

159. *To shew that*

$$\sin(-A) = -\sin A, \quad \textit{and} \ \cos(-A) = \cos A.$$



Let *PAB* be any angle; draw *PM* perpendicular to the straight line *BAB′*, and produce it to *P′* so that *MP′* may be equal in length to *MP*, and join *AP′*.

The angles *P′AB* and *PAB* are numerically equal but are measured in opposite directions from *AB*; let *PAB* be denoted by *A*, then *P′AB* will be denoted by $-A$. And

$$\sin A = \frac{PM}{AP}, \quad \sin(-A) = \frac{P'M}{AP'};$$

now *P′M* is numerically equal to *PM*, but of opposite sign: thus

$$\sin(-A) = -\sin A.$$

Also $$\cos(-A) = \frac{AM}{AP'} = \frac{AM}{AP} = \cos .$$

160. Hence we have:

$$\tan(-A) = \frac{\sin(-A)}{\cos(-A)} = \frac{-\sin A}{\cos A} = -\tan A;$$

$$\cot(-A) = \frac{\cos(-A)}{\sin(-A)} = \frac{\cos A}{-\sin A} = -\cot A;$$

$$\sec(-A) = \frac{1}{\cos(-A)} = \frac{1}{\cos A} = \sec A;$$

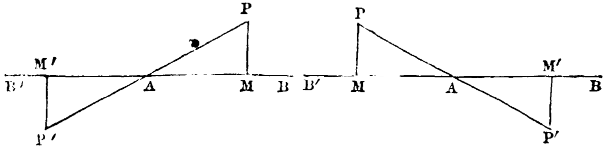
$$\operatorname{cosec}(-A) = \frac{1}{\sin(-A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A;$$

$$\operatorname{vers}(-A) = 1 - \cos(-A) = 1 - \cos A = \operatorname{vers} A.$$

These results might also be obtained directly from the figures, as in Art. 159.

161. *To shew that*

$\sin(180^0 + A) = -\sin A$, *and* $\cos(180^0 + A) = -\cos A$.

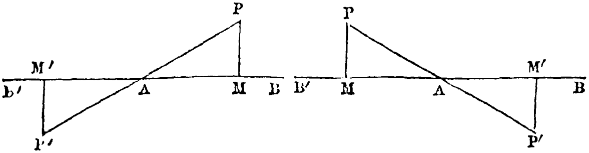


Let PAB be any angle; produce PA to P' so that AP' may be equal in length to AP: draw PM and $P'M'$, perpendicular to the straight line BAB'.

Let the angle PAB be denoted by A, then the angle $P'AB$ measured in the same direction from A will be denoted by $180^0 + A$. And

$$\sin A = \frac{PM}{AP}, \quad \sin(180^0 + A) = \frac{P'M'}{AP'},$$

$$\cos A = \frac{AM}{AP}, \quad \cos(180^0 + A) = \frac{AM'}{AP'}.$$



The triangles PAM and $P'AM'$ are geometrically equal in all respects; thus PM and $P'M'$ are numerically equal, but they are of opposite sign: also AM and AM' are numerically equal but of opposite sign. Thus

$$\sin(180^0+A)=-\sin A, \quad \cos(180^0+A)=-\cos A.$$

162. Hence we have:

$$\tan(180^0+A)=\frac{\sin(180^0+A)}{\cos(180^0+A)}=\frac{-\sin A}{-\cos A}=\tan A;$$

$$\cot(180^0+A)=\frac{\cos(180^0+A)}{\sin(180^0+A)}=\frac{-\cos A}{-\sin A}=\cot A;$$

$$\sec(180^0+A)=\frac{1}{\cos(180^0+A)}=\frac{1}{-\cos A}=-\sec A;$$

$$\operatorname{cosec}(180^0+A)=\frac{1}{\sin(180^0+A)}=\frac{1}{-\sin A}=-\operatorname{cosec} A;$$

$$\begin{aligned}\operatorname{vers}(180^0+A)=1-\cos(180^0+A)&=1+\cos A\\&=2-\operatorname{vers} A.\end{aligned}$$

These results might also be obtained directly from the figures as in Art. 161.

163. We can now demonstrate the statement made in Art. 158.

By the formulæ in Arts. 159 and 160, we can express the Trigonometrical Ratios of a *negative* angle in terms of the Trigonometrical Ratios of a *positive* angle. Thus we need only consider positive angles.

By Art. 148 any multiple of four right angles may be rejected. Thus we need only consider angles less than four right angles.

By Arts. 161 and 162 we can express the Trigonometrical Ratios of an angle between two and four right angles in terms of the Trigonometrical Ratios of an angle less than two right angles.

By Art. 95 we can express the Trigonometrical Ratios of an angle between one and two right angles in terms of the Trigonometrical Ratios of an angle less than a right angle.

Thus the statement is demonstrated.

164. It will be observed that when we thus reduce the angle we can always express a Trigonometrical Ratio of any angle in terms of the *same* Trigonometrical Ratio of the reduced angle.

165. As examples of the reduction of the angle we have:

$$\sin 700^0 = \sin(360^0 + 340^0) = \sin 340^0 = \sin(180^0 + 160^0)$$
$$= -\sin 160^0 = -\sin 20^0;$$

$$\cos(-800^0) = \cos 800^0 = \cos(720^0 + 80^0) = \cos 80^0;$$

$$\tan 500^0 = \tan(360^0 + 140^0) = \tan 140^0 = -\tan 40^0;$$

$$\cot 460^0 = \cot(360^0 + 100^0) = \cot 100^0 = -\cot 80^0;$$

$$\sec 930^0 = \sec(720^0 + 210^0) = \sec 210^0 = \sec(180^0 + 30^0)$$
$$= -\sec 30^0;$$

$$\operatorname{cosec}(-600^0) = -\operatorname{cosec} 600^0 = -\operatorname{cosec}(360^0 + 240^0)$$
$$= -\operatorname{cosec} 240^0 = -\operatorname{cosec}(180^0 + 60^0) = \operatorname{cosec} 60^0.$$

EXAMPLES. XVII.

Find the values of the following twelve Trigonometrical Ratios:

1. $\sin 225^0$. 2. $\sin 810^0$. 3. $\sin(-240^0)$.

4. $\cos 210^0$. 5. $\cos 540^0$. 6. $\cos(-300^0)$.

7. $\tan 195^0$. 8. $\tan 345^0$. 9. $\tan(-120^0)$.

10. $\cot 420^0$. 11. $\cot 510^0$. 12. $\cot(-315^0)$.

13. With the notation of Chapter XIV. shew that the area of a triangle is equal to

$$Rr(\sin A+\sin B+\sin C).$$

14. Shew also that the area is equal to

$$\tfrac{1}{2}R^2(\sin 2A+\sin 2B+\sin 2C).$$

15. From the bottom of a station in a horizontal plane the altitude of the summit of a mountain is found to be a, and on retiring c feet from the station its top is seen to be in a straight line with the top of the mountain: shew that if h be the height of the station the height of the mountain is $\dfrac{ch}{c-h\cot a}$ feet.

16. If CD subtend an angle a at each of the stations A and B, which are both on the same side of CD, and are distant h apart, and the sum of the two angles ABD and BAC be σ, shew that the distance between C and D is either

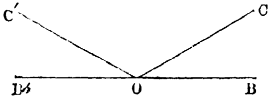
$$\frac{h\sin a}{\sin(\sigma-a)} \text{ or } \frac{h\sin a}{\sin(\sigma+a)}.$$

XVIII. *Angles with given Trigonometrical Ratios.*

166. We have shewn in Art. 17 that corresponding to a given angle there is only one value for an assigned Trigonometrical Ratio. But corresponding to a given value of an assigned Trigonometrical Ratio there is an unlimited number of angles, as we see from Chapter xv.

We shall now investigate expressions which include all the angles having a given value of an assigned Trigonometrical Ratio.

167. *To find an expression for all the angles which have a given sine.*



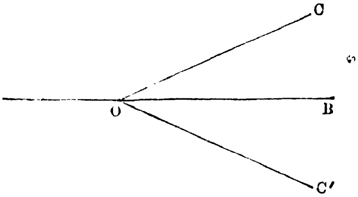
Let BOC be the least positive angle which has the given sine; denote this angle by A. Produce BO to any point B' and make the angle $B'OC' = BOC$; then $BOC' = 180^0 - A$.

Now it is obvious from the figure that the only *positive* angles which have the same sine as A are $180^0 - A$, and the angles formed by adding any multiple of four right angles to A or to $180^0 - A$; that is, angles included in the expressions $n360^0 + A$ and $n360^0 + 180^0 - A$, where n is zero or any positive integer. Also the only *negative* angles which have the same sine as A are $-(180^0 + A)$ and $-(360^0 - A)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the expressions $n360^0 - (180^0 + A)$, and $n360^0 - (360^0 - A)$, where n is zero or any negative integer.

All the angles which have been indicated will be found on trial to be included in the expression $n180^0 + (-1)^n A$, where n is zero or any integer positive or negative. Also all the angles included in this expression will be found among the angles which have been indicated.

Thus this expression includes all the angles which have the same sine as A; and all the angles which it includes have the same sine as A. This expression also applies for all the angles which have the same *cosecant* as A.

168. *To find an expression for all the angles which have a given cosine.*



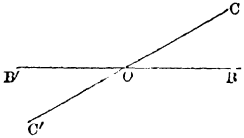
Let BOC be the least positive angle which has the given cosine; denote this angle by A. Make the angle $BOC' = BOC$.

Now it is obvious from the figure that the only *positive* angles which have the same cosine as A are $360^0 - A$, and the angles formed by adding any multiple of four right angles to A or to $360^0 - A$; that is, angles included in the expressions $n360^0 + A$ and $n360^0 + 360^0 - A$, where n is zero or any positive integer. Also the only *negative* angles which have the same cosine as A are $-A$ and $-(360^0 - A)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the expressions $n360^0 - A$, and $n360^0 - (360^0 - A)$, where n is zero or any negative integer.

All the angles which have been indicated will be found on trial to be included in the expression $n360^0 \pm A$, where n is zero or any integer positive or negative. Also all the angles included in this expression will be found among the angles which have been indicated.

Thus this expression includes all the angles which have the same cosine as A; and all the angles which it includes have the same cosine as A. This expression also applies for all the angles which have the same *secant* as A.

169. *To find an expression for all the angles which have a given tangent.*



Let BOC be the least positive angle which has the given tangent; denote this angle by A. Produce BO to any point B', and CO to any point C'.

Now it is obvious from the figure that the only *positive* angles which have the same tangent as A are 180^0+A, and the angles formed by adding any multiple of four right angles to A or to 180^0+A; that is angles included in the expressions $n360^0+A$ and $n360^0+180^0+A$, where n is zero or any positive integer. Also the only *negative* angles which have the same tangent as A are $-(180^0-A)$ and $-(360^0-A)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the expressions $n360^0-(180^0-A)$ and $n360^0-(360^0-A)$, where n is zero or any negative integer.

All the angles which have been indicated will be found on trial to be included in the expression $n180^0+A$, where n is zero or any integer positive or negative. Also all the angles included in this expression will be found among the angles which have been indicated.

Thus this expression includes all the angles which have the same tangent as A; and all the angles which it includes have the same tangent as A. This expression also applies for all the angles which have the same *cotangent* as A.

EXAMPLES. XVIII.

Give the general solutions of the following eight equations:

1. $\sin A = \frac{1}{2}$. 2. $\cos A = \frac{1}{2}$. 3. $\tan A = \sqrt{3}$.

4. $\operatorname{cosec} A = 1$. 5. $\sec A = -1$. 6. $\cot A = 2 - \sqrt{3}$.

7. $\sin A + \operatorname{cosec} A = 2$. 8. $\sin 2A = \cos 3A$.

9. Shew that the length of the straight line drawn to bisect the angle A of a triangle and terminated by the opposite side is $\frac{2bc \cos \frac{1}{2} A}{b+c}$.

10. The angle C of a triangle is a right angle. Shew that the radius of the inscribed circle is equal to

$$\frac{1}{2}\{a + b - \sqrt{(a^2 + b^2)}\}.$$

11. Shew that the radius of a circle which passes through the vertex A of a triangle, and touches the side BC at its middle point is

$$\frac{2(b^2 + c^2) - a^2}{8b \sin C}.$$

12. P, Q, and R are points in the sides BC, CA, and AB respectively of a triangle, such that

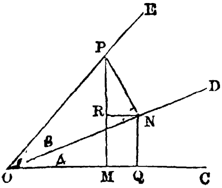
$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = x:$$

shew that $PQ^2 + QR^2 + RP^2 = (a^2 + b^2 + c^2)(1 - 3x + 3x^2)$.

XIX. *Trigonometrical Ratios of two angles.*

170. The object of the present Chapter is to express the Trigonometrical Ratios of the sum or difference of two angles in terms of the Trigonometrical Ratios of the angles themselves.

171. *To express the sine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A+B$.

In OE take any point P, draw PM perpendicular to OC, and PN perpendicular to OD; draw NR perpendicular to PM, and NQ perpendicular to OC.

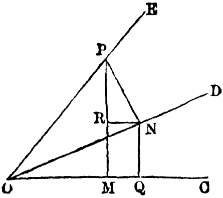
Then the angle NPR is the complement of PNR, and is therefore equal to RNO, which is equal to NOQ or A.

$$\text{Now } \sin(A+B)=\frac{PM}{OP}=\frac{RM+PR}{OP}=\frac{NQ}{OP}+\frac{PR}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}+\frac{PR}{PN}\cdot\frac{PN}{OP}$$

$$=\sin A\cos B+\cos A\sin B.$$

172. *To express the cosine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



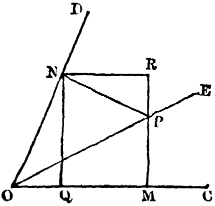
The same construction being made as in the preceding Article, we have

$$\cos(A+B)=\frac{OM}{OP}=\frac{OQ-QM}{OP}=\frac{OQ}{OP}-\frac{NR}{OP}$$

$$=\frac{OQ}{ON}\cdot\frac{ON}{OP}-\frac{NR}{NP}\cdot\frac{NP}{OP}$$

$$=\cos A\cos B-\sin A\sin B.$$

173. *To express the sine of the difference of two angles in terms of the sines and cosines of the angles themselves.*



Let the angle *COD* be denoted by *A*, and the angle *DOE* by *B*; then the angle *COE* will be denoted by *A* − *B*.

In *OE* take any point *P*, draw *PM* perpendicular to *OC*, and *PN* perpendicular to *OD*; draw *NR* perpendicular to *MP* produced, and *NQ* perpendicular to *OC*.

Then the angle NPR is the complement of PNR, and is therefore equal to DNR, which is equal to NOQ or A.

$$\text{Now } \sin(A-B)=\frac{PM}{OP}=\frac{RM-RP}{OP}=\frac{NQ}{OP}-\frac{RP}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}-\frac{RP}{PN}\cdot\frac{PN}{OP}$$

$$=\sin A\cos B-\cos A\sin B.$$

174. *To express the cosine of the difference of two angles in terms of the sines and cosines of the angles themselves.*

The same construction being made as in the preceding Article we have

$$\cos(A-B)=\frac{OM}{OP}=\frac{OQ+QM}{OP}=\frac{OQ}{OP}+\frac{NR}{OP}$$

$$=\frac{OQ}{ON}\cdot\frac{ON}{OP}+\frac{NR}{PN}\cdot\frac{PN}{OP}$$

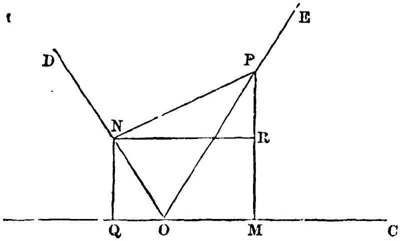
$$=\cos A\cos B+\sin A\sin B.$$

175. To assist the student in remembering the preceding demonstrations, we may observe that the point P is taken in the straight line which *bounds the compound angle we are considering;* thus in demonstrating the formulæ for $\sin(A+B)$ and $\cos(A+B)$ the point P is taken in the straight line which bounds the angle $A+B$, and in demonstrating the formulæ for $\sin(A-B)$ and $\cos(A-B)$ the point P is taken in the straight line which bounds the angle $A-B$.

176. The formulæ established in Arts. 171...174 are true whatever may be the size of the angles A and B; the student may exercise himself by going through the construction and demonstration in various cases; it will be found that the only variety which occurs in the construction consists in the circumstance that the perpendiculars in

some cases fall on certain straight lines and in other cases fall on those straight lines *produced.*

177. We will for example demonstrate the formulæ of Arts. 173 and 174, for the case in which the angle A is greater than a right angle, while B is less than a right angle, and $A-B$ is less than a right angle.



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A-B$.

In OE take any point P, draw PM perpendicular to OC and PN perpendicular to OD; draw NR perpendicular to MP and NQ perpendicular to CO produced.

Then the angle NPR is the complement of PNR, and is therefore equal to RNO, which is equal to 180^0-A. Then

$$\sin(A-B)=\frac{PM}{OP}=\frac{RM+PR}{OP}=\frac{NQ}{OP}+\frac{PR}{OP}$$

$$=\frac{NQ}{ON}\cdot\frac{ON}{OP}+\frac{PR}{PN}\cdot\frac{PN}{OP}$$

$$=\sin(180^0-A)\cos B+\cos(180^0-A)\sin B$$

$$=\sin A\cos B-\cos A\sin B, \text{ by Art. 95.}$$

And $\cos(A-B)$

$$=\frac{OM}{OP}=\frac{MQ-OQ}{OP}=\frac{NR}{OP}-\frac{OQ}{OP}$$

$$=\frac{NR}{PN}\cdot\frac{PN}{OP}-\frac{OQ}{ON}\cdot\frac{ON}{OP}$$

$$=\sin(180^0-A)\sin B-\cos(180^0-A)\cos B$$

$$=\sin A\sin B+\cos A\cos B,\text{ by Art. 95.}$$

178. By examining in this manner the various cases which can occur, the student may convince himself that the formulæ of Arts. 171...174 are universally true. Another mode of demonstration will be found in the larger work on Trigonometry, Art. 80.

We now proceed to express the tangent and cotangent of a compound angle in terms of the tangents and cotangents of the single angles.

179. $\text{Tan}(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B};$$

divide both numerator and denominator of the last expression by $\cos A\cos B$; thus we obtain

$$\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}},$$

therefore $\quad \tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}.$

180. Tan $(A-B)$

$$=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

181. Cot $(A+B)$

$$=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\dfrac{\cos A}{\sin A}\dfrac{\cos B}{\sin B}-1}{\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}.$$

182. Cot $(A-B)$

$$=\frac{\cos(A-B)}{\sin(A-B)}=\frac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$$

$$=\frac{\dfrac{\cos A}{\sin A}\dfrac{\cos B}{\sin B}+1}{\dfrac{\cos B}{\sin B}-\dfrac{\cos A}{\sin A}}=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

183. We may give other forms to these expressions. For example,

$$\cot(A+B)=\frac{1}{\tan(A+B)}=\frac{1-\tan A\tan B}{\tan A+\tan B},$$

$$\cot(A-B)=\frac{1}{\tan(A-B)}=\frac{1+\tan A\tan B}{\tan A-\tan B}.$$

184. Some important formulæ are deducible from those already given by supposing $B=A$.

Write $B=A$ in Art. 171; thus

$$\sin 2A = 2 \sin A \cos A.$$

Write $B=A$ in Art. 172; thus

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A \\ \text{or} &= 2\cos^2 A - 1. \end{aligned}$$

Thus $$1 + \cos 2A = 2\cos^2 A,$$
$$1 - \cos 2A = 2\sin^2 A,$$

and $$\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A; \text{ also } \frac{\sin 2A}{1+\cos 2A} = \tan A.$$

And since $\sin^2 A + \cos^2 A = 1$ we have

$$\sin 2A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A};$$

divide both numerator and denominator of the last expression by $\cos^2 A$; thus

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}.$$

Similarly $$\cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

In Art. 179 put $B=A$; thus

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A},$$

and this result may also be deduced from the values of $\sin 2A$ and $\cos 2A$ just given.

In Art. 181, put $B=A$; thus

$$\cot 2A = \frac{\cot^2 A - 1}{2 \cot A}.$$

EXAMPLES. XIX.

1. If $\tan A = \frac{4}{3}$ and $\tan B = 7$, find $\tan(A + B)$.

2. If $\sin A = \frac{5}{13}$ and $\sin B = \frac{7}{25}$, find $\sin(A + B)$.

3. If $\cos A = \frac{40}{41}$ and $\cos B = \frac{60}{61}$, find $\cos(A - B)$.

4. If $\tan A = \frac{1}{\sqrt{3}}$ and $\tan B = \frac{1}{\sqrt{15}}$, find $\sin(A + B)$.

5. If $\tan B = \frac{2 \sin A \sin C}{\sin(A + C)}$, shew that $\tan A$, $\tan B$, and $\tan C$ are in Harmonical Progression.

6. If $\tan A = a$ and $\tan B = b$, find $\cos 2(A + B)$ and $\sin 2(A + B)$.

7. If $\cos(A - B) = n \sin(A + B)$, shew that

$$\tan(45^0 + A) = \frac{n + 1}{n - 1} \tan(45^0 - B).$$

8. If $\sin A = \frac{1}{\sqrt{3}}$ and $\sin B = \frac{1 + \sqrt{6}}{2\sqrt{3}}$, find $\cos(A + B)$.

9. Given $\tan 2A = \frac{24}{7}$, find $\sin A$.

10. If $\tan 2A = 2 \tan B$ and $\tan C = \tan^3 A$, shew that $\tan(B - C) = \tan A$.

11. If $\cos A = \frac{p + qc}{q + pc}$ and $\cos B = \frac{p - qc}{q - pc}$, find $\cos(A + B)$.

12. If $\tan A = \tan^2 B$, then $2 \tan 2A = \sin 2B \tan 2B$.

13. Given $\sin A - \cos A = \frac{1}{\sqrt{2}} \cos 1^0 - \frac{\sqrt{3}}{\sqrt{2}} \sin 1^0$, find the number of degrees in the least value of A.

14. If $\tan B + \cot B = 2 \sec 2A$, shew that one value of $A + B$ is 45^0.

15. Shew that $\sec 2A - \tan 2A = \tan (45^0 - A)$.

16. If $\cot B - 2 \cot 2B = \sec 2A - \tan 2A$, shew that one value of $A + B$ is 45^0.

17. If $p \sin A = q \sin B$ and $r \cos A = s \cos B$, shew that $\cos (A + B) = \frac{qs - pr}{qr - ps}$, and $\cos (A - B) = \frac{qs + pr}{qr + ps}$.

Demonstrate the following ten identities:

18. $\frac{\sin 2A}{1 + \cos 2A} \cdot \frac{\cos A}{1 + \cos A} = \tan \frac{A}{2}$.

19. $\sin 8A = 8 \sin A \cos A \cos 2A \cos 4A$.

20. $(\sin A - \sin B)^2 + (\cos A - \cos B)^2 = 2 \operatorname{vers} (A - B)$.

21. $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$.

22. $2 \sin 2A - \sin 4A = 4 \sin 2A \sin^2 A$.

23. $\cot^2 A - \tan^2 A = 4 \cot 2A \operatorname{cosec} 2A$.

24. $2 - 2 \tan A \cot 2A = \sec^2 A$.

25. $\sec^2 (A + 45^0) - \sec^2 (A - 45^0) = 4 \tan 2A \sec 2A$.

26. $\tan A + \cot 2A = \operatorname{cosec} 2A$.

27. $\tan (45^0 + A) - \tan (45^0 - A) = 2 \tan 2A$.

XX. *Trigonometrical Transformations.*

185. From the formulæ of the preceding Chapter others may be derived; and by the aid of all the results we can change Trigonometrical expressions into various forms. The use of such transformations becomes apparent as the student advances in Mathematics, and even at present he will find valuable exercise in them.

186. The product of $\sin(A+B)$ and $\sin(A-B)$ takes a remarkable form.

$$\text{Sin}\,(A+B)\sin(A-B)$$

$$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$$

$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$$

$$=\sin^2 A\,(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$$

$$=\sin^2 A-\sin^2 B.$$

This may also be put in the form $\cos^2 B-\cos^2 A$.

187. Also

$$\cos(A+B)\cos(A-B)$$

$$=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$$

$$=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$$

$$=\cos^2 A\,(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$$

$$=\cos^2 A-\sin^2 B$$

$$=\cos^2 B-\sin^2 A.$$

188. From the four fundamental formulæ of Chapter XIX. we have

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B,$$

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B,$$

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B,$$

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Let $A+B=C$, and $A-B=D$; therefore

$$A=\frac{C+D}{2},\quad B=\frac{C-D}{2}:\ \text{thus}$$

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2},$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2},$$

$$\cos D-\cos C=2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

The last four formulæ are useful in converting a sum or difference into the form of a product of factors. Thus, for example, by the first of the last four formulæ the sum of two sines is converted into twice the product of a sine and cosine.

The first four formulæ are useful in converting a product into the form of a sum or difference. Thus, for example, by the last of the first four formulæ we see that the product of two sines is equal to half the cosine of the difference of the two angles diminished by half the cosine of their sum.

189. $$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}} \text{ (Art. 188)}$$

$$=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$$

$$=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}.$$

190. $$\frac{\cos A+\cos B}{\cos B-\cos A}=\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}} \text{ (Art. 188)}$$

$$=\cot\frac{A+B}{2}\cot\frac{A-B}{2}.$$

191. $$\tan A+\tan B=\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}$$

$$=\frac{\sin(A+B)}{\cos A\cos B}.$$

Similarly $$\tan A-\tan B=\frac{\sin(A-B)}{\cos A\cos B}.$$

192. $$\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\sin A\cos A}$$

$$=\frac{1}{\sin A\cos A}=\frac{2}{2\sin A\cos A}=\frac{2}{\sin 2A}.$$

193. $$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} = \frac{\cos 2A}{\sin A \cos A}$$

$$= \frac{2\cos 2A}{2\sin A \cos A} = \frac{2\cos 2A}{\sin 2A} = 2\cot 2A.$$

194. By repeated applications of the formulæ in Chapter XIX, we can obtain expressions for the Trigonometrical Ratios of a composite angle made up of any number of simple angles connected by the signs + and −. For example,

$$\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$$
$$= \sin A \cos B \cos C + \sin B \cos C \cos A$$
$$+ \sin C \cos A \cos B - \sin A \sin B \sin C.$$

$$\cos(A+B+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= \cos A \cos B \cos C - \cos C \sin A \sin B$$
$$- \cos B \sin A \sin C - \cos A \sin B \sin C.$$

$$\tan(A+B+C) = \frac{\sin(A+B+C)}{\cos(A+B+C)}:$$

substitute the expressions just found for $\sin(A+B+C)$ and $\cos(A+B+C)$; and then divide both numerator and denominator by $\cos A \cos B \cos C$; thus we obtain

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

195. The particular case of the formulæ in the preceding Article in which $C = B = A$ should be noticed. Thus we obtain

$$\sin 3A = 3\sin A \cos^2 A - \sin^3 A$$
$$= 3\sin A(1 - \sin^2 A) - \sin^3 A$$
$$= 3\sin A - 4\sin^3 A;$$

$$\begin{aligned}\cos 3A &= \cos^3 A - 3\cos A\sin^2 A \\ &= \cos^3 A - 3\cos A\,(1-\cos^2 A) \\ &= 4\cos^3 A - 3\cos A\,;\end{aligned}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

196. When angles are connected by a relation we often find some simple relation connecting some of their Trigonometrical Ratios. We will take for example the case in which there are three angles, the sum of which is equal to 180^0; this relation holds for the angles of a triangle.

If $A+B+C=180^0$, then will

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

For $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ (Art. 183)

$$\begin{aligned}&= 2\sin\left(90^0 - \frac{C}{2}\right)\cos\frac{A-B}{2} \\ &= 2\cos\frac{C}{2}\cos\frac{A-B}{2}\,;\end{aligned}$$

$$\begin{aligned}\sin C &= 2\sin\frac{C}{2}\cos\frac{C}{2}, \\ &= 2\cos\frac{C}{2}\cos\left(90^0 - \frac{C}{2}\right) \\ &= 2\cos\frac{C}{2}\cos\frac{A+B}{2}\,;\end{aligned}$$

therefore $\sin A+\sin B+\sin C$

$$=2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}$$

$$=2\cos\frac{C}{2}\,2\cos\frac{A}{2}\cos\frac{B}{2}\text{ (Art. 188)}$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Again, if $A+B+C=180^0$, then will

$$\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}.$$

For $\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$

$$=\frac{\cos\frac{B}{2}\sin\frac{C}{2}+\sin\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$=\frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\quad\text{(Art. 171)}$$

$$=\frac{\sin\left(90^0-\frac{A}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Similarly if $A+B+C=180^0$, then

$$\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}.$$

197. The solution of equations involving Trigonometrical Ratios, is facilitated by transformations of the kind given in the present Chapter. For example; required to find the value of the angle A from the equation

$$\sin 7A-\sin A=\sin 3A.$$

By Art. 188, $\sin 7A-\sin A=2\sin 3A\cos 4A$;

thus we have the equation $2\sin 3A\cos 4A=\sin 3A$;

therefore either $\sin 3A=0$, or $\cos 4A=\frac{1}{2}$.

The former gives as the simplest solution $3A=0$, and for the general solution $3A=n\ 180^0$.

The latter gives as the simplest solution $4A=60^0$, and for the general solution $4A=n\ 360^0\pm 60^0$.

See Chapter XVIII.

Again, required to find the value of the angle A from the equation $\qquad \cos 8A+\cos 4A=2\cos 2A.$

By Art. 188, $\cos 8A+\cos 4A=2\cos 6A\cos 2A$;

thus we have the equation $2\cos 6A\cos 2A=2\cos 2A$;

therefore either $\cos 2A=0$, or $\cos 6A=1$.

The former gives as the simplest solution $2A=90^0$, and for the general solution $2A=n\ 360^0\pm 90^0$; this may be also put in the form $(2m+1)\ 90^0$.

The latter gives as the simplest solution $6A=0$, and for the general solution $6A=n\ 360^0$.

EXAMPLES. XX.

Demonstrate the following thirty identities :

1. $\dfrac{\sin 3A + \sin A}{\cos 3A + \cos A} = \tan 2A.$

2. $\dfrac{\cos A - \cos 3A}{\sin A + \sin 3A} = \tan A.$

3. $\dfrac{\sin 5A - \sin 3A}{\cos 3A - \cos 5A} = \cot 4A.$

4. $\dfrac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan (A + B).$

5. $\cos^2 2A - \cos^2 3A = \sin A \sin 5A.$

6. $\sin A \sin (A + 2B) - \sin B \sin (2A + B) = \sin^2 A - \sin^2 B.$

7. $\dfrac{\sin A + \sin (A + B) + \sin (A + 2B)}{\cos A + \cos (A + B) + \cos (A + 2B)} = \tan (A + B).$

8. $2 \cos (n - 1) A \cos A - \cos (n - 2) A = \cos nA.$

9. $2 \sin \left(\dfrac{A + B}{2} - 45^0\right) \cos \left(\dfrac{A - B}{2} + 45^0\right)$

$$= \sin A - \cos B.$$

10. $(1 + \cot A + \operatorname{cosec} A)(1 + \cot A - \operatorname{cosec} A)$

$$= \cot \frac{A}{2} - \tan \frac{A}{2}.$$

11. $\dfrac{\tan 3A}{\tan A} = \dfrac{2 \cos 2A + 1}{2 \cos 2A - 1}.$

12. $\sin 3A - \cos 3A = (\sin A + \cos A)\,(4 \sin A \cos A - 1)$.

13. $\sin B + \dfrac{\sin (A - B) - \sin B}{1 + \cos A} = \tan \dfrac{A}{2} \cos B$.

14. $2 \sec (A + B)\,(\cos^2 A - \sin^2 B)$
$= (\sin 2A + \sin 2B) \operatorname{cosec} (A + B)$.

15. $2 \sin \dfrac{3A}{2} \cos \dfrac{3A}{2} \cos 2A - 2 \sin A \cos A \cos 3A$
$= \sin A$.

16. $2 (\cos^8 A - \sin^8 A) = \cos 2A\,(1 + \cos^2 2A)$.

17. $\sin (3A + B) \sin (3A - B) - \sin (A + B) \sin (A - B)$
$= \sin 4A \sin 2A$.

18. $\{\cos(A + B) + \sin(A - B)\}\{\sin(A + B) + \cos(A - B)\}$
$= \cos 2B(1 + \sin 2A)$.

19. $\dfrac{3 \sin A - \sin 3A}{3 \cos A + \cos 3A} = \left(\dfrac{\sec 2A - 1}{\sec 2A + 1}\right)^{\frac{3}{2}}$.

20. $(\cos A - \sin A)\,(\cos 2A - \sin 2A) + \sin 3A = \cos A$.

21. $(3 \sin A - 4 \sin^3 A)^2 + (4 \cos^3 A - 3 \cos A)^2 = 1$.

22. $1 - \cos 3A = (1 - \cos A)\,(1 + 2 \cos A)^2$.

23. $\tan \dfrac{A}{2} + 2 \sin^2 \dfrac{A}{2} \cot A = \sin A$.

24. $\cos A - \sin A \tan \dfrac{A}{2} = \cos 2A + \sin 2A \tan \dfrac{A}{2}$.

25. $\dfrac{2 \cos 2A}{\cos A + \sin A} + \dfrac{2 \sin 2A}{\cos A - \sin A} = \dfrac{\sqrt{2}}{\cos (A + 45^\circ)}$.

26. $\cos(A+B-C)+\cos(A-B+C)+\cos(B+C-A)$
$$+\cos(A+B+C)=4\cos A\cos B\cos C.$$

27. $\cos x+\cos y+\cos z+\cos(x+y+z)$
$$=4\cos\frac{x+y}{2}\cos\frac{y+z}{2}\cos\frac{z+x}{2}.$$

28. $\sin(A+B-C)+\sin(A+C-B)+\sin(B+C-A)$
$$-\sin(A+B+C)=4\sin A\sin B\sin C.$$

29. $\sin x+\sin y+\sin z-\sin(x+y+z)$
$$=4\sin\frac{x+y}{2}\sin\frac{y+z}{2}\sin\frac{z+x}{2}.$$

30. $\sin(A+B+C)+\sin(B+C-A)+\sin(A+C-B)$
$$-\sin(A+B-C)=4\sin C\cos A\cos B.$$

If $A+B+C=180^0$ demonstrate the following five relations:

31. $\sin 2A+\sin 2B-\sin 2C=4\sin C\cos A\cos B.$

32. $\sin\frac{A}{2}\cos\frac{B-C}{2}+\sin\frac{B}{2}\cos\frac{C-A}{2}+\sin\frac{C}{2}\cos\frac{A-B}{2}$
$$=\cos A+\cos B+\cos C.$$

33. $\sin^2 A+\sin^2 B+\sin^2 C$
$$2\sin B\sin C\cos A+2\sin A\sin C\cos B+2\sin A\sin B\cos C.$$

34. $$\frac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B}=\tan\frac{A}{2}\cot\frac{C}{2}.$$

35. $\sin 2B(1+2\cos 2C)+\sin 2C(1+2\cos 2A)$
$$+\sin 2A(1+2\cos 2B)$$
$$=4\sin(C-B)\sin(A-C)\sin(B-A).$$

XXI. *Division of Angles.*

198. By Art. 184, we have

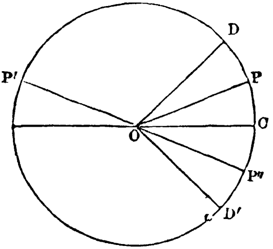
$$\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2};$$

hence $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$.

These formulæ serve to determine the sine and the cosine of half an angle, when the cosine of the angle is given. It will be seen that by reason of the double sign we have *two* values for $\sin\frac{A}{2}$ and *two* values for $\cos\frac{A}{2}$ corresponding to a given value of $\cos A$.

199. The reason why there is more than one value for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, corresponding to a given value of $\cos A$, is that corresponding to a given value of the cosine there is more than one value of the angle.

Thus suppose that the angle COD has its cosine equal to $\cos A$, then an angle equal to four right angles di-



minished by COD also has its cosine equal to $\cos A$; this angle is denoted in the figure by the larger angle bounded by OC and OD'. If we take COD for A, then $\frac{1}{2}A$ is the angle COP, where OP is such that $COP=POD$. If we take for A the larger angle bounded by OC and OD', then $\frac{1}{2}A$ is the angle COP', where OP' is such that $COP'=P'OD'$. Also the angle measured in the negative direction between OC and OD', that is the angle COD', has its cosine equal to $\cos A$. If we take COD' for A, then $\frac{1}{2}A$ is the angle COP'', where OP'' is such that $COP''=P''OD'$.

It is easy to see on examining the figure that COP' is the supplement of COP, and that OP' and OP'' are in the same straight line. Hence it follows that the sines of COP, COP' and COP'' are numerically equal but have not all the same sign; so also the cosines of COP, COP' and COP'' are numerically equal but have not all the same sign.

If any of the angles which we have taken for A be increased by any multiple of four right angles, we shall obtain an angle which has its cosine equal to $\cos A$; it will be found on examining the figure that the sine and the cosine of half such an angle will coincide respectively with the sine and the cosine of one of the angles which we have already taken for $\frac{1}{2}A$.

Thus we have accounted for the fact that $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ when expressed in terms of $\cos A$ have each two values numerically equal but of opposite signs.

200. By assuming the result obtained in Art. 168 we can put the preceding explanation into a briefer form.

All the angles which are comprised in the expression $n360^\circ \pm A$, where n is any integer, positive or negative, have the same cosine as A. Hence we may expect that any formula which gives $\sin\frac{A}{2}$ in terms of $\cos A$ will include the sine of every angle which is comprised in the expression $\frac{1}{2}(n360^\circ \pm A)$, that is in the expression $n180^\circ \pm \frac{1}{2}A$.

Now $\sin(n180^0 \pm \frac{1}{2}A) = \sin(\pm\frac{1}{2}A)$ if n be even,

and $= -\sin(\pm\frac{1}{2}A)$ if n be odd.

And thus, by Art. 159, we have the two values $\pm\sin\frac{1}{2}A$, and no more.

Similarly we may expect that any formula which gives $\cos\frac{A}{2}$ in terms of $\cos A$ will include the cosine of every angle comprised in the expression $n180^0 \pm \frac{1}{2}A$.

Now $\cos(n180^0 \pm \frac{1}{2}A) = \cos(\pm\frac{1}{2}A)$ if n be even,

and $= -\cos(\pm\frac{1}{2}A)$ if n be odd.

And thus, by Art. 159, we have the two values $\pm\cos\frac{1}{2}A$, and no more.

201. If in any case we actually know the value of A we know also the value of $\frac{1}{2}A$; and then we can settle which sign we ought to take in the formula for $\sin\frac{1}{2}A$, and which sign we ought to take in the formula for $\cos\frac{1}{2}A$. And even if we do not know the exact value of A we may know sufficient to enable us to make this selection; for example, if we know that A lies between 90^0 and 180^0, then we know that $\frac{1}{2}A$ lies between 45^0 and 90^0, and the positive sign must be taken in both the formulæ of Art.

202. Remarks similar to those which have been made in the last three Articles will be found applicable also to numerous other results in Trigonometry in which the double sign occurs; for examples we may mention the remaining results of the present Chapter, or the result $\sin A = \pm\sqrt{1-\cos^2 A}$, and others of the same kind. We shall however not enlarge on this point, for we have given enough to illustrate the general principle; more applications will be found in Chapter VII. of the larger Trigonometry.

203. By Art. 184

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2};$$

and $$1 = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2};$$

thus $$\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)^2 = 1 + \sin A,$$

and $$\left(\sin \frac{A}{2} - \cos \frac{A}{2}\right)^2 = 1 - \sin A;$$

therefore $$\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{\left(1 + \sin A\right)} \qquad (1),$$

and $$\sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{\left(1 - \sin A\right)} \qquad (2).$$

Thus as soon as the proper signs are known in (1) and (2) we can by addition and subtraction find expressions for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in terms of $\sin A$.

Let us take the case in which A is an *acute* angle, then $\frac{A}{2}$ lies between 0^0 and 45^0; thus $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ are both positive, and $\cos \frac{A}{2}$ is greater than $\sin \frac{A}{2}$. We must therefore take the upper sign in (1) and the lower sign in (2), so that

$$\sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{(1 + \sin A)},$$

$$\sin \frac{A}{2} - \cos \frac{A}{2} = -\sqrt{(1 - \sin A)};$$

therefore $$2 \sin \frac{A}{2} = \sqrt{(1 + \sin A)} - \sqrt{(1 - \sin A)},$$

and $$2 \cos \frac{A}{2} = \sqrt{(1 + \sin A)} + \sqrt{(1 - \sin A)}.$$

204. As an example of the formulæ of the preceding Article we will find the sine and the cosine of an angle of 9^0.

$$2 \sin 9^0 = \sqrt{(1 + \sin 18^0)} - \sqrt{(1 - \sin 18^0)},$$

$$2 \cos 9^0 = \sqrt{(1 + \sin 18^0)} + \sqrt{(1 - \sin 18^0)};$$

$$1 + \sin 18^0 = 1 + \frac{\sqrt{5} - 1}{4} = \frac{3 + \sqrt{5}}{4},$$

$$1 - \sin 18^0 = 1 - \frac{\sqrt{5} - 1}{4} = \frac{5 - \sqrt{5}}{4}.$$

Thus $$\sin 9^0 = \frac{1}{4} \left\{ \sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})} \right\},$$

and $$\cos 9^0 = \frac{1}{4} \left\{ \sqrt{(3 + \sqrt{5})} + \sqrt{(5 - \sqrt{5})} \right\}.$$

205. By Art. 195 we have

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3},$$

$$\sin A = 3 \sin \frac{A}{3} - 4 \sin^3 \frac{A}{3}.$$

Thus if $\cos A$ be given and we require $\cos \frac{A}{3}$ we have to solve a cubic equation; and similarly if $\sin A$ be given and we require $\sin \frac{A}{3}$ we have to solve a cubic equation.

EXAMPLES. XXI.

1. Find the cosine of $11\frac{1}{4}^0$.

2. If A be between 90^0 and 180^0, shew that

$$2\sin\frac{A}{2} = \sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}.$$

3. If $\sin 224^0 = -\cdot 69$, write down the value of $\sin 112^0$.

4. Shew that $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$. Explain the double sign.

5. Shew that $\sin A$ when expressed in terms of $\sin\frac{A}{2}$ has two equal values of opposite signs; and that $\cos A$ when expressed in terms of $\cos\frac{A}{2}$ has only one value.

6. Shew that

$$\cot\left(45^0 - \frac{A}{2}\right) - \cot\left(45^0 + \frac{A}{2}\right) = 2\tan A.$$

7. Shew that

$$\sin 5A \operatorname{cosec}^2 A \sec A - \cos 5A \sec^2 A \operatorname{cosec} A = 8\cot 2A.$$

8. Shew that

$$\frac{\cos A}{1-\cos A} - \frac{1+\cos A}{\cos A} = 2\cot 2A \cot\frac{A}{2}.$$

9. If $a\sin\theta + b\cos\theta = c = a\operatorname{cosec}\theta + b\sec\theta$,

show that $$(a^{\frac{2}{3}} - b^{\frac{2}{3}})^2 (a^{\frac{2}{3}} + b^{\frac{2}{3}}) = c^2.$$

10. If

$$\sin\theta + \sin\phi = m,\ \cos\theta + \cos\phi = n,\ \text{and } \cos(\theta+\phi) = p,$$

shew that $$\frac{m^2}{n^2} = \frac{1-p}{1+p}.$$

XXII. *Circular Measure.*

206. In *practice* angles are always estimated by means of *degrees, minutes* and *seconds;* but there is another method of estimating angles which is very important in *theory,* which we will now explain. The object of the present Chapter is to establish and apply the following proposition: *If with the point of intersection of any two straight lines as centre a circle be described with any radius, then the angle contained by the straight lines may be measured by the ratio which the length of the arc of the circle intercepted between the straight lines bears to the radius.*

207. *The circumferences of circles vary as their radii.*

If a regular polygon of any number of sides be inscribed in a circle, and a regular polygon of the *same* number of sides be inscribed in another circle, the perimeters of the polygons are as the radii of the circles. See Art. 138. This is true however great be the number of the sides; and we may assume that by making the number of sides as large as we please the perimeters of the polygons will not differ sensibly from the perimeters of the corresponding circles.

For a fuller exhibition of this demonstration the student may consult Chapter II. of the larger work on Trigonometry.

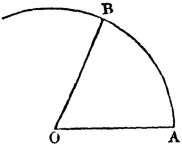
208. Thus the ratio of the circumference of a circle to its radius is *constant* whatever be the magnitude of the circle; therefore of course the ratio of the *circumference* to the *diameter* is also constant. The numerical value of the ratio of the circumference of a circle to its diameter cannot be stated exactly; but it is shewn in the larger work on Trigonometry that the ratio may be calculated to

any degree of approximation. The value is approximately equal to $\frac{22}{7}$, and still more nearly equal to $\frac{355}{113}$; the value correct to eight places of decimals is 3·14159265...

The symbol π is invariably used to denote the ratio of the circumference of a circle to its diameter; hence if r denote the radius of a circle its circumference is $2\pi r$: and

$$\pi = 3{\cdot}14159265...$$

209. *The angle subtended at the centre of a circle by an arc which is equal in length to the radius is an invariable angle.*



With centre O and any radius OA describe a circle; let AB be an arc of this circle equal in length to the radius. Then, since angles at the centre of a circle are proportional to the arcs on which they stand,

$$\frac{\text{angle } AOB}{\text{4 right angles}} = \frac{\text{arc } AB}{\text{circumference of the circle}}$$

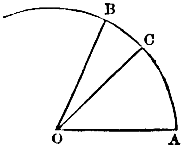
$$= \frac{r}{2\pi r} = \frac{1}{2\pi};$$

therefore $$\text{angle } AOB = \frac{\text{4 right angles}}{2\pi}.$$

Thus the angle AOB is a certain fraction of four right angles, which is constant, whatever may be the radius of the circle.

210. Since the angle subtended at the centre of a circle by an arc which is equal to the radius is an *invariable angle* it may be taken as the *unit* of angular measurement, and then any angle will be estimated by the ratio which it bears to this unit.

Let AOC be any angle; with O as centre and any radius OA describe a circle; let AB be an arc of this circle equal in length to the radius; let r denote the radius, and l the length of the arc AC.



Then, since angles at the centre of a circle are proportional to the arcs on which they stand,

$$\frac{\text{angle } AOC}{\text{angle } AOB} = \frac{AC}{AB} = \frac{l}{r};$$

therefore angle $AOC = \frac{l}{r} \times \text{angle } AOB$; this result is true whatever the unit of angular measurement may be, the same unit of course being used for the two angles. If we take the angle AOB itself for the unit, then this angle must be denoted by unity;

$$\text{thus angle } AOC = \frac{l}{r}.$$

211. We have thus shewn that any angle may be estimated by a fraction which has *for its numerator the arc which the angle intercepts on any circle having its centre at the angular point, and for its denominator the radius of that circle.* And in this mode of estimating angles the unit, that is the angle denoted by unity, is the

angle in which the arc intercepted is equal to the radius. We have shewn that this angle is $\frac{4 \text{ right angles}}{2\pi}$; hence the number of *degrees* contained in the angle is $\frac{360}{2\pi}$, that is $\frac{180}{\pi}$. If we use the approximate value of π given in Art. 208, we shall find that $\frac{180}{\pi} = 57{\cdot}29577951\ldots$; this therefore is the number of degrees contained in the angle which is subtended at the centre of a circle by an arc equal to the radius.

212. Thus there are two methods of forming an idea of the magnitude of an angle which is estimated by the fraction *arc divided by radius.* Suppose, for example, we speak of the angle $\frac{3}{4}$: we may refer to the unit of angular measurement which is an angle containing about 57 degrees, and imagine three-fourths of this unit to be taken; or without thinking about the unit at all, we may suppose that an angle is taken such that the arc subtending it is three-fourths of the corresponding radius.

The fraction *arc divided by radius* is called the *circular measure of an angle.* Since, as we have already stated, this method of measuring angles is very much used in theoretical investigations, it is sometimes called the *theoretical* method.

213. If r denote the radius of a circle the circumference is $2\pi r$; hence the circular measure of four right angles is $\frac{2\pi r}{r}$, that is 2π. The circular measure of two right angles is π; the circular measure of one right angle is $\frac{\pi}{2}$; and the circular measure of n right angles is $\frac{n\pi}{2}$, where n may be either integral or fractional.

Let x denote the number of degrees in any angle, θ the circular measure of the same angle. Since there are 180 degrees in two right angles, $\frac{x}{180}$ expresses the ratio of this angle to two right angles. And since π is the circular measure of two right angles, $\frac{\theta}{\pi}$ also expresses the ratio of the angle to two right angles.

Hence $$\frac{x}{180} = \frac{\theta}{\pi};$$

thus $$x = \frac{180\theta}{\pi},$$

and $$\theta = \frac{\pi x}{180}.$$

215. For example, the circular measure of an angle of one degree is $\frac{\pi}{180}$, the circular measure of an angle of three degrees is $\frac{3\pi}{180}$, that is $\frac{\pi}{60}$; the circular measure of an angle of one minute is $\frac{\pi}{180 \times 60}$; the circular measure of an angle of one second is $\frac{\pi}{180 \times 60 \times 60}$; and so on.

Again, if the circular measure of an angle is $\frac{2}{3}$, the number of degrees contained in the angle is $\frac{2}{3} \cdot \frac{180}{\pi}$; that is $\frac{2}{3}$ of 57·29577951...; if the circular measure of an

The student who intends to proceed to the higher parts of mathematics is recommended to pay particular attention to the points illustrated by these examples; especially he should accustom himself to express readily in circular measure an angle which is given in degrees.

216. Similarly we may connect the circular measure of any angle with the measure of the same angle in *grades.*

Let y denote the number of grades in any angle, θ the circular measure of the same angle. Since there are 200 grades in two right angles, $\frac{y}{200}$ expresses the ratio of this angle to two right angles. And since π is the circular measure of two right angles, $\frac{\theta}{\pi}$ also expresses the ratio of the angle to two right angles.

Hence $$\frac{y}{200}=\frac{\theta}{\pi};$$

thus $$y=\frac{200\theta}{\pi},$$

and $$\theta=\frac{\pi y}{200}.$$

The number of grades in the angle which is the unit of circular measure is $\frac{200}{\pi}$, that is $63\cdot661977\ldots$.

For example, the circular measure of an angle of ten grades is $\frac{10\pi}{200}$, that is $\frac{\pi}{20}$. Again, if the circular measure of an angle is $\frac{4}{5}$, the number of grades contained in the angle is $\frac{4}{5}\cdot\frac{200}{\pi}$; that is $\frac{4}{5}$ of $63\cdot661977\ldots$

EXAMPLES. XXII.

Express the following three angles in circular measure:

1. $\frac{5}{8}$ of a right angle. 2. $24^0\,30'$. 3. 50^g.

Find the number of degrees in the following three angles which are expressed in circular measure:

4. $2\frac{1}{2}$. 5. $\cdot 6$. 6. $\cdot 375$.

7. If the length of an arc of a circle which subtends 3 degrees at the centre be 6 feet, find the radius, and the length of an arc subtending 3 grades.

8. Find the number of degrees in the angle subtended at the centre by an arc of 7 feet when the radius is 10 feet.

9. The radius of a circle is 20 feet: determine the length of an arc which subtends an angle of $30'$ at the centre.

10. The circular measure of the difference of the two acute angles of a right-angled triangle is $\frac{\pi}{18}$: express the two angles in degrees.

11. The angles of a triangle are in Arithmetical Progression; and the circular measure of the common Difference is $\frac{\pi}{9}$: determine the angles.

12. Find the circular measure of the angle which is the excess of m degrees above n grades.

13. Compare the angle $a^0\,b'$ with the angle $a^g\,b^{\backprime}$.

14. If n be any whole number, shew that $\frac{n}{200}$ of a right angle contains a whole number both of English minutes and of French minutes.

15. What must be the unit angle if the sum of the measures of a degree and a grade is 1?

16. An angle is made up of two parts, one containing a degrees and the other containing b grades: compare the angle with a right angle.

17. If $A+B+C=180^0$, shew that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

18. If the tangents of the three angles of a triangle be as the numbers 1, 2, 3, shew that the tangents must be equal to 1, 2, 3.

19. Having given $\tan(a+\theta)\tan(a-\theta)=k$, find $\sin\theta$.

20. If $\cos A\cos 3A=\frac{5}{18}$, then $\sin^2 A=\frac{1}{6}$.

21. If $\cos(s-2a)+\cos(s-2\beta)=\cos(s-2\gamma)+\cos(s-2\delta)$,

where $s=a+\beta+\gamma+\delta$,

then $\tan a\tan\beta=\tan\gamma\tan\delta$.

22. Shew that

$$2\cos 5^0\,37'\,30''=\sqrt{[2+\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}]}.$$

23. If in a triangle $3\tan\frac{C}{2}=\cot\frac{A}{2}$, shew that the sides are in Arithmetical Progression.

24. Shew that the triangle in which

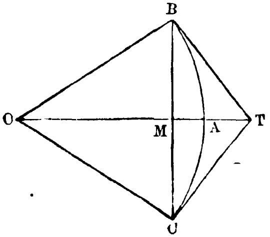
$$\frac{5-4\cos A}{5-4\cos C}=\frac{\sin^2 A}{\sin^2 C}$$

is either isosceles, or such that the sides are in Arithmetical Progression.

XXIII. *Area of a Circle.*

217. The principal object of the present Chapter is to find an expression for the area of a circle; we shall give some important preliminary propositions.

218. *If θ be the circular measure of a positive angle less than a right angle, θ is greater than $\sin\theta$ and less than $\tan\theta$.*



Let AOB be an angle less than a right angle, and let $OB=OA$; from B draw BM perpendicular to OA and produce it to C so that $MC=MB$; draw BT at right angles to OB meeting OA produced at T, and join CT.

Then the triangles MOC and MOB are equal in all respects, so that the angle $TOC=$ the angle TOB; therefore the triangles TOC and TOB are equal in all respects, so that TCO is a right angle, and $TC=TB$.

With centre O and radius OB describe an arc of a circle BAC; this will touch BT at B and CT at C.

Now we assume as an axiom that the straight line BC is less than the arc BAC; thus BM, the half of BC, is less than BA, the half of the arc BAC; therefore $\frac{BM}{OB}$ is less than $\frac{BA}{OB}$; that is, the sine of AOB is less than the circular measure of AOB.

Again, we assume as an axiom that the arc BAC is less than the sum of the two exterior straight lines BT and TC; thus BA is less than BT; therefore $\frac{BA}{OB}$ is less than $\frac{BT}{OB}$; that is, the circular measure of AOB is less than the tangent of AOB.

Hence $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude if θ be less than $\frac{\pi}{2}$.

219. *The limit of* $\frac{\sin\theta}{\theta}$ *when* θ *is indefinitely diminished is unity.*

For $\sin\theta$, θ, and $\tan\theta$ are in ascending order; divide by $\sin\theta$; therefore 1, $\frac{\theta}{\sin\theta}$, and $\frac{1}{\cos\theta}$ are in ascending order of magnitude. Thus $\frac{\theta}{\sin\theta}$ lies in value between 1 and $\frac{1}{\cos\theta}$; but when θ is zero, $\cos\theta$ is unity; hence as θ diminishes indefinitely $\frac{\theta}{\sin\theta}$ approaches the limit unity. Therefore also $\frac{\sin\theta}{\theta}$ approaches the limit unity. And as $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}$, the limit of $\frac{\tan\theta}{\theta}$ when θ is indefinitely diminished is also unity.

220. *To find the area of a circle.*

Let r be the radius of a circle. Suppose a regular polygon of n sides described about the circle. Then the circular measure of the angle which each side subtends at the centre of the circle is $\frac{2\pi}{n}$; and therefore, by Art. 139, the area of the polygon is $nr^2 \tan\frac{\pi}{n}$.

$$\text{Now } nr^2 \tan\frac{\pi}{n} = \frac{\pi r^2}{\cos\frac{\pi}{n}} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}.$$

Suppose n to increase without limit, then the area of the polygon approximates continually to the area of the circle as the limit, and therefore the area of the circle will be equal to the limit of the above expression.

But when n is indefinitely great

$$\cos\frac{\pi}{n} = 1, \quad \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1\,; \text{ (Art. 219)}$$

therefore the *area of a circle of radius* $r = \pi r^2$.

221. *To find the area of a sector of a circle.*

Let θ be the circular measure of the angle of the sector; then, by Euclid VI. 33,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi};$$

$$\text{therefore area of sector} = \pi r^2 \times \frac{\theta}{2\pi} = \frac{r^2\theta}{2}.$$

Thus the area of a sector is equal to *half the product of the square of the radius into the circular measure of the angle.*

Since θ is the circular measure of the angle the length of the arc of the sector is $r\theta$; hence the area of a sector is equal to *half the product of the length of the arc into the radius.*

EXAMPLES. XXIII.

1. With the angular points of an equilateral triangle as centres, and a radius equal to half the side, arcs are described touching each other: determine the area of the figure which they form.

2. A chord of length r is placed in a circle of radius r: determine the areas of the two segments into which the chord divides the circle.

3. A circle is described round a triangle the angles of which are 45°, 60°, and 75° respectively: determine the areas of the segments of the circle cut off by the sides.

4. Two circles touch each other, and a common tangent is drawn. Supposing their radii to be r and $3r$ respectively, shew that the area of the curvilinear triangle bounded by the two circles and the common tangent is

$$\left(4\sqrt{3}-\frac{11\pi}{6}\right)r^2.$$

5. From the formula $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, shew that $\cos\theta$ is greater than $1-\frac{\theta^2}{2}$.

6. If $\frac{\cos\theta}{\theta}+\frac{\theta}{\cos\theta}$ has its least possible arithmetical value, shew that θ is greater than $\sqrt{3}-1$.

7. Shew by Arts. 103 and 188 that in any triangle

$$c=(a-b)\frac{\cos\frac{C}{2}}{\sin\frac{A-B}{2}}=(a+b)\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}.$$

8. The vertical angle of a triangle is 120°, and the difference of the sides is equal to $\frac{4}{9}$ of the base: find the other angles.

$L \sin 12^\circ\, 50' = 9{\cdot}3465794$, $\log 2 = {\cdot}3010300$,

$L \sin 12^\circ\, 51' = 9{\cdot}3471336$, $\log 3 = {\cdot}4771213$.

9. One of the angles of a plane triangle is 60°, and the side opposite is to the difference of the two sides including it as 9 is to 2: find the other angles.

$L \cos 78^\circ\, 54'\, 10'' = 9{\cdot}2843730$, $\log 3 = {\cdot}4771213$,

$L \cos 78^\circ\, 54'\, 20'' = 9{\cdot}2842656$.

10. In a triangle $a = 19$, $b = 1$, $A - B = 90^\circ$: find C.

$L \tan 41^\circ\, 59' = 9{\cdot}9541834$, $\log 3 = {\cdot}4771213$,

$L \tan 42^\circ = 9{\cdot}9544374$.

11. A person standing in the same plane with two vertical poles, and at a distance from the nearer equal to the distance a between them, sees their summits in the same direction. After walking in a straight horizontal line b feet towards the nearer pole he observes that the altitude of one summit is double that of the other. Determine the heights of the two summits.

12. From the deck of a ship which is sailing due North a lighthouse is observed due East, and the altitude of its summit is found to be $12^\circ\, 26'$. After the ship has sailed ten miles the lighthouse is again observed, and its altitude is found to be $7^\circ\, 17'$. Determine how far the lighthouse was distant from the ship at the first observation.

$L \sin 5^\circ\, 9' = 8{\cdot}9530996$, $L \sin 7^\circ\, 17' = 9{\cdot}1030373$,

$L \sin 19^\circ\, 43' = 9{\cdot}5281053$, $L \sin 77^\circ\, 34' = 9{\cdot}9896932$,

$\log 7{\cdot}1142 = {\cdot}8521261$, $\log 7{\cdot}1143 = {\cdot}8521322$.

XXIV. *Inverse Notation.*

222. The equation $\sin x = a$ asserts that x is an angle of which the sine is a; it is found convenient to be able to express this relation also in another way, in which x stands alone. The notation used is this, $x = \sin^{-1} a$. Similarly $x = \cos^{-1} a$ expresses that x is an angle of which the cosine is a; and $x = \tan^{-1} a$ expresses that x is an angle of which the tangent is a; and so on.

223. Any relation which has been established among Trigonometrical Ratios may be expressed by means of the inverse notation. Thus, for example, we know that

$$\cos 2\theta = 2\cos^2\theta - 1;$$

this may be written

$$2\theta = \cos^{-1}(2\cos^2\theta - 1):$$

suppose that $\cos\theta = a$, so that $\theta = \cos^{-1} a$,

thus $$2\cos^{-1} a = \cos^{-1}(2a^2 - 1).$$

Again, we know that

$$\sin 2\theta = 2\sin\theta\cos\theta;$$

this may be written

$$2\theta = \sin^{-1}(2\sin\theta\cos\theta):$$

suppose that $\sin\theta = a$, so that $\theta = \sin^{-1} a$, and $\cos\theta = \sqrt{1 - a^2}$,

thus $$2\sin^{-1} a = \sin^{-1}(2a\sqrt{1 - a^2}).$$

224. Also any relation which is expressed in the inverse notation may be converted into a relation expressed in the ordinary notation. Thus, for example, suppose we have given that

$$2\tan^{-1} a = \tan^{-1}\frac{2a}{1 - a^2};$$

take the tangents of both sides; thus

$$\tan(2\tan^{-1} a) = \frac{2a}{1 - a^2}:$$

suppose that $\tan^{-1}a=\theta$, so that $a=\tan\theta$,

thus
$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}.$$

225. As an example of the inverse notation suppose we require the value of $\sin(\sin^{-1}a+\cos^{-1}b)$.

Let $\sin^{-1}a=A$, and $\cos^{-1}b=B$; then the proposed expression becomes

$$\sin(A+B) \text{ or } \sin A\cos B+\cos A\sin B,$$

now
$$\sin A=a, \quad \cos A=\sqrt{(1-a^2)},$$
$$\cos B=b, \quad \sin B=\sqrt{(1-b^2)};$$

therefore
$$\sin(\sin^{-1}a+\cos^{-1}b)=ab+\sqrt{(1-a^2)}\sqrt{(1-b^2)}.$$

For a numerical illustration take $a=\frac{1}{2}$, and $b=\frac{1}{2}$;

therefore
$$\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right)=\frac{1}{4}+\frac{3}{4}=1.$$

We may express this relation also thus,

$$\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\sin^{-1}1.$$

Since $\sin\frac{\pi}{2}=1$ we have as *one* possible result,

$$\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\frac{\pi}{2}.$$

Examples, like the result just given, are often proposed for exercise; but it should be remembered that $\sin^{-1}\frac{1}{2}$ and $\cos^{-1}\frac{1}{2}$ both have an infinite number of values, and thus $\frac{\pi}{2}$ is merely *one* out of an infinite number of possible values of the left-hand member. See Chapter XVIII.

EXAMPLES. XXIV.

1. If $\sin\theta = a$, express in the inverse notation
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

2. If $\tan\theta = a$, express in the inverse notation
$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

3. Shew that $\cos(2\tan^{-1} a) = \dfrac{1 - a^2}{1 + a^2}$.

4. Express without inverse notation
$$\sin^{-1}\tan\frac{\pi}{4} \text{ and } \tan\sin^{-1}\frac{1}{\sqrt{2}}.$$

5. Shew that $\tan^{-1}\dfrac{1}{2} + \operatorname{cosec}^{-1}\sqrt{10} = \dfrac{\pi}{4}$.

6. Shew that $2\tan^{-1}\dfrac{2}{3} - \operatorname{cosec}^{-1}\dfrac{5}{3} = \sin^{-1}\dfrac{33}{65}$.

7. In any right angled triangle, in which C is the right angle,
$$\cot^{-1}\sqrt{\frac{c+a}{c-a}} + \cot^{-1}\sqrt{\frac{c+b}{c-b}} = \frac{\pi}{4}.$$

8. Shew that $2\tan^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{4}{5} = \dfrac{\pi}{2}$.

9. Shew that
$$\sin^{-1}(\cos x) + \cos^{-1}(\sin y) + x + y = \pi.$$

10. If $\tan^{-1}3x - \tan^{-1}x = \tan^{-1}\dfrac{1}{2}$, find x.

11. If $\tan^{-1}\dfrac{a+x}{a} + \tan^{-1}\dfrac{a-x}{a} = \dfrac{\pi}{6}$, shew that
$$x^2 = 2a^2\sqrt{3}.$$

12. If $\dfrac{\pi}{2} = \sin^{-1}x + \tan^{-1}x$, shew that
$$x^2 = \frac{\sqrt{5} - 1}{2}.$$

MISCELLANEOUS EXAMPLES.

1. Express three-tenths of a right angle in degrees and in grades.

2. If $\tan A = \frac{a}{b}$ find $\sin A$ and $\cos A$.

3. Shew that $\tan 2A + \cos A \operatorname{cosec} A = \cot A \sec 2A$.

4. Solve the equation $\tan A + \cot A = 4$.

5. Solve the equation $\frac{\cot A}{\cos 2A} - \frac{\operatorname{cosec} A}{\sec A} = 1$.

6. If $\cos A = \frac{19}{20}$ find $\cos 4A$.

7. The town C is half-way between the towns D and E; and the towns C, E, and F are equidistant from each other: compare the distance of D from F with the distance of D from E.

8. Find the area of a triangle in which the length of the base is c, and the angles at the base 15^0 and 75^0.

9. Determine the logarithm of 32 to the base 16, and the logarithm of 16 to the base 32.

10. Given $\log 67663 = 4{\cdot}8303512$, $\log 67664 = 4{\cdot}8303577$, find $\log 67663{\cdot}2$.

11. If the unit of measurement be 5^0, find the measure of $22\frac{1}{2}^0$.

12. One of the angles of a quadrilateral contains 60 degrees, another contains 50 grades, and another is equal to three-fourths of two right angles: express all the angles in degrees.

13. Shew that

$$(\sin A + \cos A)^3 + (\sin A - \cos A)^3 = 2 \sin A\,(3 - 2 \sin^2 A).$$

14. Shew that $\tan 2A - \sec A \sin A = \tan A \sec 2A$.

15. Solve the equation $\sin A + \operatorname{cosec} A = 2$.

16. Solve the equation $\cot^2 A - \tan^2 A = 2 \sec A \operatorname{cosec} A$.

17. If A and B are acute angles, and A the greater, shew that $\frac{\tan A}{\tan B}$ is greater than $\frac{\sin A}{\sin B}$.

18. Find the height of a tower whose top appears at an elevation of 30^0 to an observer 120 feet from the foot of the tower, on a horizontal plane, his eye being 5 feet from the ground.

19. Find the characteristic of the logarithm of 1593 to the base 10, and also to the base 12.

20. The hypotenuse of a right-angled triangle is 580 feet; and the tangent of one of the angles is $\frac{20}{21}$: find the sides of the triangle.

21. Express in grades and parts of a grade the angle $24^0\,40'\,21''\cdot36$.

22. Express in each system of angular measurement the angle described by the long hand of a clock between 24 minutes past twelve o'clock and 24 minutes to one o'clock.

23. Shew that $\frac{\operatorname{cosec} A}{\sec A} + \frac{\sec A}{\operatorname{cosec} A} = \sec A \operatorname{cosec} A$.

24. Shew that $\cot^2 A - \tan^2 A = \frac{4 \cos 2A}{\sin^2 2A}$.

25. Solve the equation $\sin A \cos A = \frac{1}{2\sqrt{2}}$.

26. Solve the equation

$$3\cos^2 A - \sin^2 A + (\sqrt{3}+1)(1-2\cos A) = 0.$$

27. The diagonals of a quadrilateral are in length a and b respectively: shew that the area of the quadrilateral cannot be greater than $\frac{1}{2}ab$.

28. Given $\tan A = \frac{1}{5}$ find $\tan 2A$.

29. Given $\log 2 = \cdot 3010300$ find $\log 25$ and $\log \cdot 125$.

30. Given $\log 28952 = 4\cdot 4616786$, $\log 28953 = 4\cdot 4616936$, find the number which has the logarithm $3\cdot 461\,6870$.

31. Express in degrees and decimals of a degree the angle $24^{g}\ 67^{\backprime}\ 20^{\backprime\backprime}$.

32. Shew that

$$\cos A\,(2\sec A + \tan A)(\sec A - 2\tan A) = 2\cos A - 3\tan A.$$

33. Shew that

$$\cos^2 A + \sin^2 A\cos 2B = \cos^2 B + \sin^2 B\cos 2A.$$

34. Solve the equation $\tan A + \cot A = \dfrac{4}{\sqrt{3}}$.

35. Solve the equation $\dfrac{\tan A}{\cos 2A} + \dfrac{\sec A}{\operatorname{cosec} A} = 1$.

36. ACB is a triangle, right-angled at C, and the tangent of ABC is $\frac{3}{4}$; the side CA is produced to a point D such that the angle DBA is equal to the angle ABC: shew that the tangent of CBD is $\frac{24}{7}$.

37. Determine the altitude of the sun when the length of the shadow of a vertical stick is to the length of the stick as 1 is to $\sqrt{3}$.

38. If a, b, c d be perpendiculars from the angles of a quadrilateral on the diagonals, and h, k be the diagonals; shew that the sine of the angle between the diagonals is

$$\sqrt{\frac{(a+c)(b+d)}{hk}}.$$

39. Given $\log 5 = \cdot 6989700$, find $\log 8$ and $\log \cdot 034$.

40. The sides of a triangle are 13, 37 and 40: find the length of the perpendicular on the longest side from the opposite angle, and the sine of the least angle.

41. If one angle is nineteen times as large as another shew that their difference contains as many grades as their sum contains degrees.

42. Shew that

$$\cos A (\tan A + 2)(2 \tan A + 1) = 2 \sec A + 5 \sin A.$$

43. Shew that

$$\sin^2 A - \cos^2 A \cos 2B = \sin^2 B - \cos^2 B \cos 2A.$$

44. Find the greatest value of $\cos^4 x - \sin^4 x$.

45. Solve the equation

$$\cot 2A + 2 \tan 2A = 4 \sec 2A \operatorname{cosec} 2A \tan^2 2A - \tan^3 2A.$$

46. Given $a^2 \sin\theta \cos\theta = xy$, and $x^3 \sin\theta + y^3 \cos\theta = axy$, shew that $\frac{x^3}{y^3} = \cot^3\theta$ or $\tan\theta$.

47. Shew that the perpendicular drawn from any point in the circumference of a circle on a chord is a mean proportional between the perpendiculars drawn from the same point on the tangents at the extremities of the chord.

48. Given the difference of the lengths of the shadows of a vertical rod when the sun's altitude is α and β, find the height of the rod. Also find the relation between α and β when the height is a mean proportional between the lengths of the shadows.

49. Find log 343 to the base 7, log 2 to the base 512, and log $\frac{1}{100}$ to the base 10.

50. Find L sin 18° 4′ 42″, having given

$$L \sin 18^0\, 4'\, 40'' = 9{\cdot}4917926,\ L \sin 18^0\, 4'\, 50'' = 9{\cdot}4918571.$$

51. One angle of a triangle exceeds the difference of the other two by 60 degrees, and exceeds the smaller of the other two by 50 grades: find the angles.

52. Eliminate θ and ϕ from the equations

$$x = a \cos^m \theta \cos^m \phi,\ y = b \cos^m \theta \sin^m \phi,\ z = c \sin^m \theta.$$

53. Find the simplest value of x from the equation

$$\sin 4x = \cos 5x.$$

54. Find sin x from the equation

$$4 \sin x + 3 \cos x = 5.$$

55. A ladder 20 feet long leaning against a column reaches to a point 20 feet from the top. From the foot of the ladder the angle of elevation of the column is 60°. Find the height of the column.

56. Shew that $2 \sin 11\frac{1}{4}^0 = \sqrt{2 - \sqrt{2 + \sqrt{2}}}$.

57. In a triangle $\sin^2 C = \sin^2 A + \sin^2 B$: shew that the triangle is right-angled.

58. An equilateral triangle, a square, and a regular hexagon have their perimeters equal: shew that their areas are nearly in the proportion of 10, 13, and 15.

59. Given $\log 5 = {\cdot}6989700$, $\log 24 = 1{\cdot}3802112$, find log 3, log 75 and log $\sqrt{75}$.

60. Find A if $L \sin A = 9{\cdot}6518969$, having given

$$L \sin 26^0\, 39'\, 20'' = 9{\cdot}6518843,\ L \sin 26^0\, 39'\, 30'' = 9{\cdot}6519263.$$

61. The difference of two angles of an isosceles triangle is 20 grades: determine all the angles in degrees.

62. Find the greatest value of $\sin x \cos x$.

63. If $\sin x + \cos x = \sin A + \cos A$, shew that $\sin x$ must be equal to $\sin A$ or to $\cos A$.

64. Shew that $2 \sin 33\tfrac{3}{4}^\circ = \sqrt{2 - \sqrt{2 - \sqrt{2}}}$.

65. Solve the equation $\sec^2 \dfrac{A}{2} + \operatorname{cosec}^2 \dfrac{A}{2} = 8 \cot A$.

66. If $\cot \alpha + \cos \phi = \sqrt{2} \cot \theta \sin \phi$,

nd $\cot \beta + \cos \theta = \sqrt{2} \cot \phi \sin \theta$;

hew that
$$\frac{\sin^2 \alpha}{\sin^2 \beta} = \frac{\sin^2 \theta}{\sin^2 \phi}.$$

67. If the angles of a triangle are in the proportion of 2, 3, find the proportion of the sides.

68. If $\dfrac{\cos A}{2 \cos B - 1} = \dfrac{1}{2 - \cos B}$, shew that
$$\tan^2 \frac{A}{2} = 3 \tan^2 \frac{B}{2}.$$

69. Find log 1728, log 17·28 and log ·001728, having n $\log 2 = ·3010300$, $\log 3 = ·4771213$.

70. Find $L \cos 64^\circ\, 40'\, 16''$, having given

cos $64^\circ\, 40'\, 10'' = 9·6312813$, $L \cos 64^\circ\, 40'\, 20'' = 9·6312368$.

71. The number of degrees in one of the acute angles right-angled triangle is three-fifths of the number of des in the other acute angle: express the angles in rees.

2. If $\dfrac{\cos A}{\cos B} = p$, and $\dfrac{\tan A}{\tan B} = q$, find $\sin^2 A$ and $\sin^2 B$.

3. Shew that $\dfrac{1 + \sin 2A}{1 + \cos 2A} = \dfrac{1}{2}(1 + \tan A)^2$.

4. If $\tan x + \cot x = \tan A + \cot A$, shew that $\tan x$ be equal to $\tan A$ or to $\cot A$.

75. Having given $\operatorname{cosec}^4 A - \sec^4 A = 2 \sec^2 A \operatorname{cosec}^2 A$, find $\cos 2A$.

76. In a triangle $\tan B = 1$, $\tan C = 2$, $b = 100$: shew that $a = 60\sqrt{5}$, and that $\tan A = 3$.

77. The tangents of the angles of a certain triangle are 1, 2, 3: if p_1, p_2, p_3 be the perpendiculars on the sides from the opposite angles, shew that $5p_1p_2p_3 = 3abc$.

78. Two spectators, at two given stations, observe at the same time the altitude of a kite, and find it to be the same, a, at the two stations. The angle, β, which the straight line joining one station and the kite subtends at the other station is also observed; and the distance, c, between the stations is known. Find the height of the kite.

79. Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, find $\log \frac{27}{32}$ and $\log \frac{2}{375}$.

80. Find A if $L \cos A = 9\cdot 5198092$, having given

$$L \cos 70^0\,40'\,10'' = 9\cdot 5198512,\quad L \cos 70^0\,40'\,20'' = 9\cdot 5197912.$$

81. Express in degrees the complement of the angle which contains $56^g\ 5^{\backprime}\ 50^{\backprime\backprime}$.

82. If $\sin A + \sin B = p$, and $\cos^2 A - \cos^2 B = q$, find $\sin A$ and $\sin B$.

83. Shew that

$$(\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A \operatorname{cosec} A)^2.$$

84. Shew that $\dfrac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \dfrac{A}{2}$.

85. If $\cos x + \sec x = \cos A + \sec A$, shew that $\cos x$ must be equal to $\cos A$.

86. A person observes the elevation of a tower to be 15^0; he then walks 100 yards directly towards the tower, and observes that the elevation is 30^0: find the height of the tower.

87. If $\tan\phi = \cos\theta \tan\alpha$ and $\tan\beta = \tan\theta \sin\phi$, shew that $\cos^2\alpha = \cos^2\beta \cos^2\phi$.

88. Find the least value of $a^2 \tan^2 x + b^2 \cot^2 x$, where a and b are constant quantities.

89. Find log 33, log 1029, and $\log(\cdot 00231)^{\frac{1}{3}}$, having given $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$, $\log 11 = 1\cdot 0413927$.

90. Find $L \tan 18^0 39' 44''$, having given
$L \tan 18^0 39' 40'' = 9\cdot 5285632$, $L \tan 18^0 39' 50'' = 9\cdot 5286327$.

91. If the angular unit were the angle of a regular pentagon, find by what number a right angle would be denoted.

92. Find the value of $a \cos 2\theta + b \sin 2\theta$ when $\tan\theta = \frac{b}{a}$.

93. Shew that

$$(\sin A + \cos A)^4 + (\sin A - \cos A)^4 = 3 - \cos 4A.$$

94. In a right-angled triangle CD and CE are drawn from the right angle C to the hypotenuse, on the same side of the perpendicular from C on the hypotenuse, making angles α and β with the hypotenuse, of which β is the greater: shew that the area of the triangle CDE is $\frac{a^2b^2}{2c^2}(\cot\alpha - \cot\beta)$.

95. Having given $8 \cos^4\theta - 8 \cos^2\theta + 1 = 0$, find the value of $\cos 2\theta$.

96. Shew that

$$\tan A (\tan 2A)^{\frac{1}{2}} (\tan 4A)^{\frac{1}{4}} = \frac{4 \sin^2 A}{(2 \sin 8A)^{\frac{1}{4}}}.$$

97. Find the least value of $a^2 \sec^2 x + b^2 \cos^2 x$, where a and b are constant quantities.

98. An isosceles triangle of vertical angle 2α is held in a vertical plane, and faces south as the sun crosses the meridian. If β be the elevation of the sun, and 2γ the angle of the shadow on the horizontal plane, shew that $\tan\gamma = \tan\beta \tan\alpha$.

99. Find approximately the value of x from the equation $\left(\frac{10}{3}\right)^{x+2}=9^{2x-1}$; having given $\log 3=\cdot 4771213$.

100. Find A if $L\cot A=9\cdot 7728743$, having given

$L\cot 59^\circ 20' 30''=9\cdot 7728887$, $L\cot 59^\circ 20' 40''=9\cdot 7728407$.

101. Shew that $\tan^2 A-\sin^2 A=\sin^4 A\sec^2 A$.

102. Shew that

$$(\sec A+p\operatorname{cosec} A)^2-(\tan A+p\cot A)^2=(p-1)^2+4p\operatorname{cosec} 2A.$$

103. Solve the equation $\sin A\cos A=\frac{1}{4}$.

104. Given $\log 2=\cdot 3010300$, $\log 3=\cdot 4771213$, find $\log 27$, $\log 36$, $\log 54$, $\log \cdot 0025$, and $\log\frac{10}{81}$.

105. Find the area of a triangle whose sides are 5, 6, 5 inches respectively.

106. Standing straight in front of one corner of a house I find that its length subtends an angle whose tangent is 2, while its height subtends an angle whose tangent is $\frac{3}{5}$; the height of the house is 45 feet: find the length of the house.

107. If $b=2\cdot 25$, $c=1\cdot 75$, $A=54^\circ$, find B and C. Having given $\log 2=\cdot 3010300$, $L\cot 27^\circ=10\cdot 2928341$, $L\tan 13^\circ 47'=9\cdot 3897244$, $L\tan 13^\circ 48'=9\cdot 3902700$.

108. Shew that $\sec 72^\circ-\sec 36^\circ=\sec 60^\circ$.

109. In the ambiguous case, a, b and A being given, if c_1 and c_2 are the third sides of the two triangles, c_1 being greater than c_2, shew that the distance between the centres of their circumscribing circles is $\frac{c_1-c_2}{2\sin A}$.

110. Trace the changes in the sign and the magnitude of $\cot A - \tan A$ as A changes from 0 to 90^0.

111. Shew that

$$(\sec A - \operatorname{cosec} A)(1 + \cot A + \tan A) = \frac{\sec^2 A}{\operatorname{cosec} A} - \frac{\operatorname{cosec}^2 A}{\sec A}.$$

112. Given $3 \cos A + \sin A = \sqrt{10}$, find $\cot A$.

113. Given $\sec A \operatorname{cosec} A + 2 \cot A = 4$, find $\sin 2A$.

114. Find the area of a triangle whose sides are 6, 7, 11 inches respectively.

115. Standing opposite to the stern of a barge, which is moored parallel to the bank of a stream, I find that its length subtends an angle of 45^0; on walking 100 feet along the bank I pass its bow, and then observe that its length subtends an angle whose tangent is $\frac{1}{3}$: find the length of the barge.

116. If $b = 65$, $c = 55$, $A = 63^0$, find B and C.

Having given $L \cot 31^0 30' = 10{\cdot}2126807$,

$$\log 2 = {\cdot}3010300, \quad \log 3 = {\cdot}4771213,$$

$$L \tan 7^0 44' = 9{\cdot}1328926, \quad L \tan 7^0 45' = 9{\cdot}1338391.$$

117. Shew that $\tan 36^0 = \sqrt{5} \tan 18^0$.

118. In the triangle ABC the straight line joining A to the middle point of BC is at right angles to AC: shew that $\cos A \cos C = \dfrac{2(c^2 - a^2)}{3ac}$.

119. The sides of a triangle are respectively 31, 24, and 11: determine the greatest angle.

120. Shew that there is always a value of A less than 90^0 which satisfies the equation $\cot A - \tan A = p$, whatever may be the value of p.

121. Find the French measures of the angles whose English measures are $22^\circ\,30'$, $39^\circ\,36'$, and $76^\circ\,24'\,36''$.

122. Shew that

$$\sin^3 A + \cos^3 A + \sec^3 A + \operatorname{cosec}^3 A$$
$$= (\sin A + \cos A)(1 - \sin A \cos A)(1 + \sec^3 A \operatorname{cosec}^3 A).$$

123. Solve the equation

$$2\sin^2 A + \sin^2 2A = 2.$$

124. Given $\sin A + \cos A = \frac{1}{2}$, find $\sin A$.

125. The sides of a triangle are 5, 7, 8 feet respectively: find the cosine of each angle: find also the radius of the circle which circumscribes the triangle.

126. A is the foot of a vertical pole; B and C are due East of A, and D is due South of C. The elevation of the pole at B is double that at C, and the tangent of the angle subtended by AB at D is $\frac{1}{5}$. If $BC = 20$ feet, and $CD = 30$ feet, find the height of the pole.

127. A tower surmounted by a flag-staff stands on a level plain. From a certain point in the plain the tower is observed to subtend an angle β, and the flag-staff an angle α. From a second point c feet nearer to the base of the tower the flag-staff is found again to subtend an angle α. Shew that the height of the tower is $\dfrac{c\tan\beta}{1 - \tan(\alpha+\beta)\tan\beta}$.

128. If each of the angles A and B of a triangle is double of the third angle C, shew that

$$\cos\frac{A+B}{2}\cos\frac{A+B+C}{5} = \cos^4\frac{A+B+C}{4}.$$

129. If S_1, S_2, S_3, S_4 be the areas of the four triangles whose sides are b, c, d; c, d, a; d, a, b; a, b, c; respectively, shew that $\dfrac{S_1^2 - S_2^2}{a^2 - b^2} + \dfrac{S_3^2 - S_4^2}{c^2 - d^2} = \dfrac{S_2^2 - S_3^2}{b^2 - c^2} + \dfrac{S_4^2 - S_1^2}{d^2 - a^2}$.

130. A triangular field has its sides respectively 50, 60, 70 yards in length: find its area, having given

$$\log 2 = \cdot 3010300, \qquad \log 1\cdot 469 = \cdot 1670218,$$
$$\log 3 = \cdot 4771213, \qquad \log 1\cdot 470 = \cdot 1673173.$$

131. Shew that

$$(\cos A - \sin A)(\operatorname{cosec} A - \sec A) = \sec A \operatorname{cosec} A - 2.$$

132. Given $\sin A \tan A = \frac{5}{6}$, find $\cos A$.

133. Given $\cos 2A - \sin A = \frac{1}{2}$, find $\sin A$.

134. From O the centre of the circle described round an equilateral triangle a straight line OD is drawn at right angles to the plane of the triangle, and equal to a side: determine the cosine of the angle between the straight lines which join D to any two corners of the triangle.

135. At the top P of a tower, of height h, the angles of depression of two objects A, B on a horizontal plane upon which the tower stands are $45^0 - a$ and $45^0 + a$ respectively; P, A, and B are in the same vertical plane: shew that $AB = 2h \tan 2a$.

136. Calculate the value of $\cos 45^0$ to 7 places of decimals; and the value of $\cos 22\frac{1}{2}^0$ and of $\sin 22\frac{1}{2}^0$ to 3 places of decimals.

137. A lighthouse stands 9 miles N. of a port from which a yacht sails in a direction E.N.E. until the lighthouse is N.W. of her, when she tacks and sails towards the lighthouse until the port is S.W. of her, when she tacks again and sails into port. Shew that the length of the course is about 16 miles.

138. If $\cos A = \tan B$, $\cos B = \tan C$, and $\cos C = \tan A$, shew that $\sin A$, $\sin B$, and $\sin C$ are all numerically equal to $2 \sin 18^0$.

139. Find the other angles in a triangle when $A = 6^0\ 37'\ 24''$ and $3b = 7c$. Having given

$$\log 2 = \cdot 3010300, \qquad L \tan 8^0\ 13'\ 50'' = 9\cdot 1603083,$$
$$L \tan 3^0\ 18'\ 42'' = 8\cdot 7624080, \qquad L \tan 8^0\ 14' = 9\cdot 1604569.$$

140. If A, B, and C are the angles of a triangle, shew that $\sin^2 A + 2\sin B \sin C \cos A = \sin^2 B + \sin^2 C$.

141. A quadrilateral figure is inscribed in a circle; if the product of the tangents of the angles of the quadrilateral is unity, shew that the sum of two of the angles will be three times the sum of the remaining two.

142. Given $\sin A + \sin B = p$, and $\sin^3 A + \sin^3 B = q$, find $\sin A$ and $\sin B$.

143. Find $\tan x$ from the equation

$$(ap - bq)\sin x + (aq + bp)\cos x = \sqrt{(a^2+b^2)(p^2+q^2)}.$$

144. Shew that in any triangle $\sin A + \sin B$ is greater than $\sin C$.

145. A gun is fired from a fort A, and the interval between seeing the flash and hearing the report is m seconds at a station B, and n seconds at a station C; a point D is in the same straight line with BC, at a known distance a from A: shew that if $BD = b$, and $CD = c$, the velocity of sound is $\left\{\frac{(b-c)(a^2-bc)}{bn^2-cm^2}\right\}^{\frac{1}{2}}$. Examine the case in which $a^2 = bc$.

146. A man standing on a plain observes a row of equidistant pillars, the tenth and seventeenth of which subtend the same angles as they would if they stood in the position of the first and were respectively one-half and one-third of the height: shew that, neglecting the height of the eye, the line of the pillars is inclined to the straight line drawn to the first at an angle whose cosine is nearly $\frac{5}{13}$.

147. In the *ambiguous case* shew that the circles which pass respectively through the middle points of the sides of the two triangles are equal, and that they have a common chord equal to half the common side of the triangles.

148. A staff $2h$ feet high placed on the top of a tower subtends an angle α at a place whose horizontal distance from the foot of the tower is c feet: determine the height of the tower.

149. The sides a, b, c of a triangle are in Arithmetical Progression: shew that the area

$$=\frac{b}{4}\sqrt{3(3b-2a)(2a-b)}.$$

150. Trace the changes in the sign and the magnitude of $\sec A-\cos A$ as A changes from 0 to 180°.

151. Solve the equation

$$\sin A+\cos A=2\sqrt{2}\sin A\cos A.$$

152. Given $\log 75{\cdot}6=1{\cdot}8785218$, find $\log{\cdot}756$ and $\log({\cdot}0756)^{\frac{1}{3}}$.

153. Solve the triangle in which

$$a=\sqrt{3}-1,\quad b=\sqrt{3}+1,\quad A=15^0.$$

154. In a plane triangle the angle $A=45^0$, the angle $B=10^0$, and the side $c=200$ feet: find the side a.

Having given

$$\log 2={\cdot}3010300,\qquad \log 172{\cdot}64=2{\cdot}2371414,$$
$$L\sin 55^0=9{\cdot}9133645,\qquad \log 172{\cdot}65=2{\cdot}2371666.$$

155. In a plane triangle ABC, where $AB=3{\cdot}02943$, $AC=1$, and the angle $ABC=19^0$, find the other parts.

Having given $\qquad \log 3{\cdot}02943={\cdot}4813608,$

$$L\sin 19^0=9{\cdot}5126419,\qquad L\sin 80^0\,30'=9{\cdot}9940027.$$

156. If $\tan A=\frac{b}{a}$ shew that

$$\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}=\frac{2\cos A}{\sqrt{\cos 2A}}.$$

157. If $\frac{\sin\theta}{x}=\frac{\cos\theta}{y}$, and $\frac{\cos^2\theta}{x^2}+\frac{\sin^2\theta}{y^2}=\frac{10}{3(x^2+y^2)}$, find $\tan\theta$.

158. If D, E, F be the middle points of the sides of a triangle opposite the angles A, B, C respectively, shew that

$$AD^2 + BE^2 + CF^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

159. Find the tangent of the angle subtended at the centre of a circle of radius r, by a chord of length c.

160. Two objects P and Q are observed from stations A and B; AP, AQ make angles α, β respectively with the straight line BA produced; BP, BQ make angles α', β' with the same straight line: shew that the area of the triangle PAQ is

$$AB^2 \times \frac{\sin\alpha' \sin\beta' \sin(\beta - \alpha)}{2\sin(\alpha - \alpha')\sin(\beta - \beta')}.$$

161. Eliminate θ from the equations

$$a\sin\theta + b\cos\theta = h, \qquad a\cos\theta - b\sin\theta = k.$$

162. Shew that in any triangle

$$\frac{b^2 - c^2}{a^2}\sin 2A + \frac{c^2 - a^2}{b^2}\sin 2B + \frac{a^2 - b^2}{c^2}\sin 2C = 0.$$

163. AB is any chord of a circle, P is any point on the circumference of the circle, and PM is perpendicular to AB: shew that $AP \,.\, BP$ varies as PM.

164. Shew that in any triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{b^2 - c^2}{a\sin(B - C)}.$$

165. The diagonals AD, BC of a rectangle $ABDC$ meet at E. If the side $AB = a$, and $AC = b$, find $\tan AEB$ in terms of a and b.

166. The sides a, b, c of a triangle are as the numbers 4, 5, 6: find the angle B. Having given

$$L\cos 27^0\, 53' = 9{\cdot}9464040, \qquad L\cos 27^0\, 54' = 9{\cdot}9463371,$$
$$\log 2 = {\cdot}3010300.$$

167. If $b = 14$, $c = 11$, $A = 60^0$, find B and C.

Having given

$$\log 2 = \cdot 3010300, \qquad L \tan 11^0 \, 44' = 9\cdot 3174299,$$
$$\log 3 = \cdot 4771213, \qquad L \tan 11^0 \, 45' = 9\cdot 3180640.$$

168. The hypotenuse AB of a right-angled triangle ABC is trisected at the points D and E: shew that if CD and CE be drawn the sum of the squares on the sides of the triangle $CDE = \frac{2c^2}{3}$.

169. If any two sides of a triangle be bisected by straight lines from the opposite angles, shew that the distance of the point of intersection from the angle A is

$$\frac{1}{3}\sqrt{(a^2 + 4bc \cos A)}.$$

170. Find A, B, and C from the equations

$$\cos (A + B - C) = \frac{1}{2}, \quad \cos (A - B + C) = \frac{\sqrt{3}}{2},$$
$$\cos (A + B) = \sin C.$$

171. Solve the equation

$$\cos A - \sin A = 2\sqrt{2} \sin A \cos A.$$

172. Solve the equation $8^x . 125^{2-x} = 2^{4x+3} . 5^x$, having given $\log 2 = \cdot 3010300$

173. Find $\sin x$ from the equation

$$\tan x = \cos x.$$

174. Solve the triangle ABC in which the side $BC = 200$, the side $BA = 250$, and the angle $BCA = 45^0$.

Having given

$$\log 2 = \cdot 3010300, \qquad L \sin 34^0 \, 27' = 9\cdot 7525750,$$
$$\log 3\cdot 4757 = \cdot 5410423, \qquad L \cos 10^0 \, 33' = 9\cdot 9925957,$$
$$\log 3\cdot 4758 = \cdot 5410548.$$

175. If $a = 210$, $b = 110$, $C = 34^\circ\ 42'\ 30''$, find A and B. Having given

$\log 2 = \cdot 3010300$, $L \cot 17^\circ\ 21'\ 15'' = 10\cdot 5051500$.

176. Two ships sail at the same time from the same port, and sail for 5 hours at the respective rates of 8 and 10 knots an hour in straight lines inclined to each other at an angle of 60°. They then sail directly towards each other. Find the inclination of their new course to their original courses. Having given

$\log 3 = \cdot 4771213$, $L \tan 10^\circ\ 53'\ 36'' = 9\cdot 2843180$.

177. From the angle A of a triangle ABC a straight line AD is drawn bisecting the side BC at D: if b, c, A be given determine the tangent of BDA.

178. If in a triangle the feet of the perpendiculars from two angles on the opposite sides be equally distant from the middle points of these sides the other angle will be 60° or 120°, or else the triangle will be isosceles.

179. If x, y, z be the lengths of the straight lines bisecting the angles of a triangle, and terminated by the opposite sides a, b, c: shew that

$$(b+c)^2 \frac{x^2}{bc} + (c+a)^2 \frac{y^2}{ac} + (a+b)^2 \frac{z^2}{ab} = (a+b+c)^2.$$

180. If the sum of two sides of a triangle is p times the base shew that the product of the tangents of half the angles of the base is $\dfrac{p-1}{p+1}$.

181. Given $\log 5 = \cdot 6989700$, $\log 7 = \cdot 8450980$, find $\log 4$, $\log 3\cdot 5$, $\log \cdot 0014$, and $\log \dfrac{28}{125}$.

182. Find $\tan 2x$ if $a \sin x + b \cos x = \sqrt{(a^2 + b^2)}$.

183. Solve the equation $\dfrac{1 + \tan A}{1 - \tan A} = \dfrac{3}{2} \sec 2A$.

184. The sides of a triangle are 3, 5, 7 inches respectively: find the angle contained by the two smaller sides, and the area of the triangle.

185. **Shew that in any plane triangle**

$$(a+b+c)\sin\frac{1}{2}A = 2a\cos\frac{1}{2}B\cos\frac{1}{2}C.$$

186. **If $a=65$, $b=16$, $C=60^0$, find the other angles.**

Having given

$\log 3 = \cdot 4771213$, $L\tan 46^0\,20' = 10\cdot 0202203$,
$\log 7 = \cdot 8450980$, $L\tan 46^0\,21' = 10\cdot 0204732$.

187. Shew trigonometrically that if an angle of a triangle be bisected the segments of the base made by the bisecting straight line will be in the ratio of the sides of the triangle.

188. The length of the straight line which bisects the angle C of a triangle and meets the base is h; and p and q are the distances of the point where it meets the base from the angles A and B respectively. Shew that

$$hc_1^2\sec\frac{C}{2} = 2(ap^2+bq^2).$$

189. A hill is inclined to the horizon at an angle α; a straight road is carried up the hill in a direction making an angle β with the intersection of the hill and the horizon: if γ be the inclination of the road to the horizon shew that

$$\sin\gamma = \sin\alpha\sin\beta.$$

190. Two stations A and B are chosen, and four points P, Q, R, S are observed: it is found that the angles which the directions of these points as seen from A make with BA produced are 60^0, 150^0, 240^0, 300^0; the angles which the directions as seen from B make with BA are 30^0, 120^0, 270^0, 330^0. Shew that the area enclosed by $PQRS$ is

$$AB^2\times\frac{9\sqrt{3}}{4}.$$

191. Two sides of a triangle are to one another as 3 is to 4; and the cosines of the opposite angles are as 3 is to 2: find the cosine of the angle which these sides include.

192. In the triangle ABC the angle $B = 50^0$, the angle $C = 20^0$, and the side $BC = 500$ feet: find the side AC.

Having given

$\log 5 = \cdot 6989700$, $\qquad L \cos 20^0 = 9\cdot 9729858$,
$\log 4\cdot 0760 = \cdot 6102342$, $\qquad L \cos 40^0 = 9\cdot 8842540$,
$\log 4\cdot 0761 = \cdot 6102448$.

193. Calculate the area of a triangular field whose sides measure 471 yards, 406 yards, and 635 yards.

Having given

$\log 7\cdot 56 = \cdot 8785218$, $\qquad \log 1\cdot 21 = \cdot 0827854$,
$\log 2\cdot 85 = \cdot 4548449$, $\qquad \log 9\cdot 55235 = \cdot 9801100$,
$\log 3\cdot 5 = \cdot 5440680$.

194. Solve the equation $2 + \cot^2 A = 3 \sec^4 A - \tan^2 A$.

195. At noon a column in the E. S. E. cast on the ground a shadow the extremity of which was in the direction N. E.; the angle of elevation of the column being a, and the distance of the extremity of the shadow from the column c, determine the height of the column.

196. Eliminate θ from the equations

$$a \tan \theta + b \sec \theta = c, \qquad a' \cot \theta + b' \operatorname{cosec} \theta = c'.$$

197. Shew that log a to the base b has always the same sign as $(a-1)(b-1)$.

198. The sides AB, BC, CD, DA of a quadrilateral figure inscribed in a circle are in a Geometrical Progression of which the common ratio is r: shew that

$$\frac{\cos ABC}{\cos BCD} = \frac{2r^2(r^2+1)}{(r^2-1)(r^4+1)}, \text{ and } \frac{\sin ABC}{\sin BCD} = \frac{2r^2}{r^4+1}.$$

199. The sides of a church tower c feet square front due N., E., S., and W. A man on the top of the tower at its S. W. angle observes the angle of depression, a, of a railway train due S. of him, and then walking to the S. E. angle he waits until the train is due S. E. of him, when he finds that its angle of depression is β. If the train is travelling in a N. E. direction, shew that the height of the tower in feet is $\dfrac{c}{\cot a - \sqrt{2} \cot \beta}$.

200. If a, b, c are in Geometrical Progression, and $\log_c a$, $\log_b c$, $\log_a b$ in Arithmetical Progression, shew that the common difference of the Arithmetical Progression is $\frac{3}{2}$.

201. One of the angles of a plane triangle is 120^0, and the sides including it are in the ratio of 4 to 1: shew that the cotangents of the other angles are $3\sqrt{3}$ and $\frac{\sqrt{3}}{2}$.

202. Shew that

$$\frac{\sin 3A + \cos 3A}{\sin 3A - \cos 3A} = \frac{2\sin 2A + 1}{2\sin 2A - 1} \times \tan (45^0 - A).$$

203. Shew that

$$\sin A \cos (B - C) - \sin B \cos (A - C) = \sin (A - B) \cos C.$$

204. If $\cos A = \sqrt{\frac{2}{3}}$, and $\cos B = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}}$, shew that $\sin (A - B) = \frac{1}{2}$.

205. Find all the values of A between 0 and 180^0 which satisfy the equation $\sin 4A = \frac{1}{2}$.

206. Shew that $\tan^{-1}\frac{4}{3} + \tan^{-1} 7 = 135^0$.

207. An arc of a circle whose radius is 7 inches subtends an angle of $15^0\ 39'\ 6''$: find what angle an arc of the same length subtends in a circle whose radius is 2 inches.

208. Solve the equation $\sin 5\theta - \cos 3\theta = \sin \theta$.

209. If $A + B + C$ is equal to $(2n + 1)\ 180^0$, or to $(4n + 1)\ 90^0$, where n is zero or any integer, then will

$$(\sin A + \cos A)(\sin B + \cos B)(\sin C + \cos C)$$
$$= 2 \sin A \sin B \sin C + 2 \cos A \cos B \cos C + 1.$$

210. If a, b, c, d be the sides of a convex quadrilateral in which a circle can be inscribed, the area of the quadrilateral will be $\sqrt{abcd} \sin \omega$, where 2ω is the sum of two opposite angles.

211. If $\cos(\alpha-\beta)\sin(\gamma-\delta)=\cos(\alpha+\beta)\sin(\gamma+\delta)$, then $\cot\delta=\cot\alpha\cot\beta\cot\gamma$.

212. If $A+B+C=90^0$, then

$$\frac{\cos A+\sin C-\sin B}{\cos B+\sin C-\sin A}=\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}.$$

213. If $A+B=90^0$, then $\sin(A-B)=-\cos 2A$, and $\sin 2A+\sin 2B=2\cos(A-B)$.

214. Shew that

$$\cos 9A+3\cos 7A+3\cos 5A+\cos 3A=8\cos^3 A\cos 6A.$$

215. Find an expression for all the values of θ which satisfy the equation $\sin 4\theta=\sin\theta$.

216. There is an angle whose tangent is twice its sine: find the length of the arc subtending it in terms of the radius.

217. Solve the equation $\cos 3\theta-\cos 5\theta=\sin\theta$.

218. If $\sin 2A=\tan^2 x$, and $2A$ is not greater than 90^0, then $2\cos A\cos x=1+\sqrt{\cos 2x}$.

219. If $A+B+C$ is equal to $2n.180^0$ or to $(4n-1)\,90^0$, where n is zero or any integer, then will

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)$$
$$=2\sin A\sin B\sin C+2\cos A\cos B\cos C-1.$$

220. If r and ρ be the radii of two circles, and if $2s$ and 2σ be the perimeters, and S and Σ the areas of the triangles whose sides touch the circles, shew that

$$s\sigma(r-\rho)-r\sigma^2=r^2\rho,$$

and

$$S^2(r-\rho)^2-\Sigma^2(r+\rho)^2=4r^3\rho^3;$$

r, s, S being respectively greater than ρ, σ, Σ.

221. If $A+B+C=180^0$,

$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}=\frac{\sin B}{\sin A}.$$

222. Shew that

$$\tan(A+30^0)\tan(A-30^0)=\frac{1-2\cos 2A}{1+2\cos 2A}.$$

223. Shew that

$$1+\cos(2A-2B)\cos 2B=\cos^2 A+\cos^2(A-2B).$$

224. If A, B, C are the angles of a triangle, shew that $\cos\frac{A}{2}+\cos\frac{B}{2}$ is greater than $\cos\frac{C}{2}$.

225. Shew that

$$\cos(15^0-a)\sec 15^0-\sin(15^0-a)\operatorname{cosec} 15^0=4\sin a.$$

226. The driving-wheel of a locomotive engine 6 feet in diameter makes 2 revolutions in a second: find approximately the number of miles which the train passes over in an hour.

227. Find the circular measure of an angle of $7^0\ 12'$.

228. If n be any positive integer, shew that $n\sin\theta$ is numerically greater than $\sin n\theta$.

229. A given loop of string is formed into a number of regular polygons successively: shew that the polygon which has the greatest number of sides has the greatest area.

230. If a triangle be such that it is possible to draw a straight line AD meeting BC at D, so that the angle BAD is one-third of the angle BAC, and also BD one-third of BC, shew that $a^2b^2=(b^2-c^2)(b^2+8c^2)$.

231. Shew that

$$\cos(36^0+A)\cos(36^0-A)+\cos(54^0+A)\cos(54^0-A)=\cos 2A.$$

232. Solve the equation

$$\tan(x+45^0)+\tan(x-45^0)=2\tan 60^0.$$

233. If $\sin x=\sin a\sin(x+y)$, then

$$\tan x=\frac{\sin a\sin y}{1-\sin a\cos y}.$$

234. If A, B, C are the angles of a triangle, shew that $\sin\frac{A}{2}+\sin\frac{B}{2}$ is greater than $\cos\frac{C}{2}$.

235. If $A+B+C=180^0$, shew that

$$\frac{\cot A+\cot B}{\tan A+\tan B}+\frac{\cot B+\cot C}{\tan B+\tan C}+\frac{\cot C+\cot A}{\tan C+\tan A}=1.$$

236. Shew that

$$(\tan 2A-\tan A)(\sec A+\sec 3A)=2\sin A\sec A\sec 3A.$$

237. Given $\cos 36^0=\frac{\sqrt{5}+1}{4}$, and $\cos 60^0=\frac{1}{2}$, find $\cos 24^0$ to three places of decimals.

238. The lengths of three straight lines OA, OB, OC are in the proportion of 6, 3, 2; the straight lines are so placed that OB and OC are at right angles to OA on opposite sides of it: shew that the angle BAC is equal to 45^0.

239. If β, γ be two different values of θ which satisfy the equation $\frac{1}{a}\cos\theta+\frac{1}{b}\sin\theta=\frac{1}{c}$, then will

$$a\cos\frac{\beta+\gamma}{2}=b\sin\frac{\beta+\gamma}{2}=c\cos\frac{\beta-\gamma}{2}.$$

240. Having given

$$a^2\cos\alpha\cos\beta+a(\sin\alpha+\sin\beta)+1=0,$$

and $$a^2\cos\alpha\cos\gamma+a(\sin\alpha+\sin\gamma)+1=0,$$

shew that $$a^2\cos\beta\cos\gamma+a(\sin\beta+\sin\gamma)+1=0,$$

and that $$\cos\alpha+\cos\beta+\cos\gamma=\cos(\alpha+\beta+\gamma),$$

β and γ being unequal, and less than π.

241. If $\tan x=\cos\alpha\tan y$, then

$$\tan(y-x)=\frac{\tan^2\frac{\alpha}{2}\sin 2y}{1+\tan^2\frac{\alpha}{2}\cos 2y}.$$

242. Shew that

$$\sin^2(A+B)+\cos^2(A-B)=1+\sin 2A\sin 2B.$$

243. Shew that

$$\cos^2 5A=\cos 4A\cos 6A+\sin^2 A.$$

244. If A B, C are the angles of an acute-angled triangle, shew that $\sin\frac{A}{2}+\sin\frac{B}{2}$ is greater than $\sin\frac{C}{2}$.

245. Shew that

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}+\cos^{-1}\frac{56}{65}=90^0.$$

246. Two straight lines being drawn from the centre of a circle cut off an arc which is to the whole circumference as 13 is to 27: find the angle between the straight lines.

247. On the sides of a parallelogram equilateral triangles are described without the parallelogram, and a quadrilateral is formed by joining the vertices of the triangles: express the squares of the diagonals of this quadrilateral, having given the sides a and b of the parallelogram, and its area S.

248. If A, B, C be the angles of a triangle, and x, y, z any real quantities satisfying the equation

$$\frac{y\sin C-z\sin B}{x-y\cos C-z\cos B}=\frac{z\sin A-x\sin C}{y-z\cos A-x\cos C},$$

then

$$\frac{x}{\sin A}=\frac{y}{\sin B}=\frac{z}{\sin C}.$$

249. Shew that if θ is less than $\frac{\pi}{2}$ then $\tan\theta$ is greater than $2\tan\frac{\theta}{2}+2\tan^3\frac{\theta}{2}$.

250. Assuming that $\tan\theta$ is greater than θ, where θ is less than $\frac{\pi}{2}$, shew that $\tan\theta$ is greater than $\theta+\frac{\theta^3}{4}$.

251. Express according to the French measure $22^0\,42'\,9''$ and $65^0\,27'\,9''$.

252. Given $\log 2 = \cdot3010300$ and $\log 3 = \cdot4771213$, find $\log 324$ and $\log \cdot015$.

253. Two straight roads which cross one another meet a canal at angles of 30^0 and 60^0 respectively. If it be three miles by the longer of the two roads from the crossing to the canal, find the distance by the shorter. If there be a foot-path which goes the shortest way to the canal, find the distance by it.

254. Given $a = 200$, $B = 45^0$, $C = 20^0$: find b.

$$\log 2 = \cdot3010300, \quad \log 156\cdot04 = 2\cdot1932359,$$
$$L \sin 65^0 = 9\cdot9572757, \quad \log 156\cdot05 = 2\cdot1932638.$$

255. From the top of a hill I observe two cottages in a straight horizontal line before me. I find their angles of depression to be 45^0 and 30^0 respectively, and know them to be 176 yards apart. Find the height of the hill.

256. Shew that in any triangle

$$\frac{a - b\cos C}{ab^2} + \frac{b - c\cos A}{bc^2} + \frac{c - a\cos B}{ca^2} = \frac{a^4 + b^4 + c^4}{2a^2b^2c^2}$$

and that $(a\sin A + b\sin B + c\sin C)^2$

$$= (a^2 + b^2 + c^2)(\sin^2 A + \sin^2 B + \sin^2 C).$$

257. Shew that if the sides a, b, c of a triangle are in Arithmetical Progression

$$\frac{\sin(C - B)}{\sin(C + A)} = \frac{(3c + a)(c - a)}{2a(c + a)}.$$

258. A water-wheel whose diameter is 12 feet makes 30 revolutions per minute: find approximately the number of miles per hour traversed by a point in the circumference of the wheel.

259. Solve the equation

$$\sin 3x = 2\sin 2x - \sin x.$$

260. Shew that

$$\tan^{-1}\frac{\sqrt{3} + 1}{\sqrt{3} - 1} - \tan^{-1}\frac{1}{\sqrt{3}} = 45^0.$$

261. If $\tan 2A = \dfrac{2n\sqrt{(1-n^2)}}{1-2n^2}$, find $\sin A$.

262. Given $b=65$, $c=35$, $A=85^0$, find B and C.

$$\log 3 = \cdot 4771213, \quad L \tan 18^0\, 7' = 9\cdot 5147766,$$
$$L \cot 42^0\, 30' = 10\cdot 0379475, \quad L \tan 18^0\, 8' = 9\cdot 5152039.$$

263. Given $c=\sqrt{60}$, $b=\sqrt{120}$, $B=135^0$: find A and a.

264. If in a triangle

$$\sin A : \sin C :: \sin (A-B) : \sin (B-C),$$

then the squares of the sides are in Arithmetical Progression.

265. Shew with the notation of Chapter XIV. that

$$\frac{r_1-r_2}{ar_1-br_2} = \frac{r_2-r_3}{br_2-cr_3} = \frac{r_3-r_1}{cr_3-ar_1} = \frac{1}{s}.$$

266. Shew that

$$\cos 3A\,(2+3\cot A-\tan^2 A)+\sin 3A\,(2+3\tan A-\cot^2 A)$$
$$=2\,(\cos A-\sin A).$$

267. If $2\sec\theta = \sec(\theta+2a)+\sec(\theta-2a)$, shew that

$$\cos^2\theta = 2\cos^2 a.$$

268. Solve the equation $\sin x + \cos x = 1$.

269. Taking the diameter of the Earth as 8000 miles, find the angle subtended at the Earth's centre by an arc of the Equator 500 miles long.

270. Shew that

$$\frac{1}{2}\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{1}{3} = 45^0.$$

271. Express in degrees, in grades, and in circular measure the difference between a degree and a grade.

272. The sides of a triangle are $m+n$, $m-n$, and $\sqrt{2(m^2+n^2)}$, and the sine of one angle is $\frac{\sqrt{5}-1}{4}$: find the other angles.

273. Given $a=15$, $b=35$, $B=120^0$: find A.

$\log 3 = \cdot 4771213$, $L \sin 21^0\,47' = 9\cdot 5694883$,
$\log 14 = 1\cdot 1461280$, $L \sin 21^0\,48' = 9\cdot 5698043$.

274. Given $a=38$, $b=11$, $C=60^0$: find A and B.

$\log 3 = \cdot 4771213$, $L \tan 43^0\,39'\,50'' = 9\cdot 9797376$.
$\log 7 = \cdot 8450980$, $L \tan 43^0\,39'\,40'' = 9\cdot 9796954$.

275. A person walking along a straight road watches two spires until they appear in the same straight line, and finds that this straight line makes an angle β with the road. From the spot where this is the case he walks a distance c in yards, when the nearest spire lies in a direction at right angles to that of the road, and he observes that at this point the angle subtended by the two spires is a. Shew that the distance between the spires in yards is

$$\frac{c \cos a}{\cos (a+\beta)} - \frac{c}{\cos \beta}.$$

276. The sides of a triangle are in Arithmetical Progression and the area is four-fifths of the area of an equilateral triangle having the same perimeter: shew that the sides are as the numbers 7, 10, 13.

277. Shew that

$$\frac{\tan (45^0 + A)}{\tan (45^0 - A)} = \frac{2 \cos A + \sin A + \sin 3A}{2 \cos A - \sin A - \sin 3A}.$$

278. Solve the equation

$$\sin^2 x + \sin x = \cos^2 x + \cos x.$$

279. In a plane triangle the lengths of two sides are 169 and 125 yards respectively, and the included angle $= 2 \sin^{-1} \frac{33}{65}$: find the third side and the area.

280. Shew that $\sin\left(\tan^{-1}\frac{3}{4}+\cot^{-1}\frac{5}{12}\right)=\frac{63}{65}$.

281. The angles of a triangle are in the proportion of 2, 3, and 5: express the angles in grades and degrees.

282. Solve the equation

$$\sqrt{3}\tan^2\theta-(1+\sqrt{3})\tan\theta+1=0.$$

283. Given $a=8$, $b=7$, $A=120^0$: find B and C.

$$\begin{array}{ll} \log 7 = \cdot 8450980, & L\sin 60^0 = 9\cdot 9375306, \\ \log 8 = \cdot 9030900, & L\sin 49^0\, 16' = 9\cdot 8795287, \\ & L\sin 49^0\, 17' = 9\cdot 8796375. \end{array}$$

284. Given $b=50$, $c=30$, $A=30^0$: find B and C.

$$\begin{array}{ll} \log 2 = \cdot 3010300, & L\tan 43^0 = 9\cdot 9696559, \\ L\cot 15^0 = 10\cdot 5719475, & L\tan 43^0\, 1' = 9\cdot 9699091. \end{array}$$

285. Given $\log 2=\cdot 3010300$, and $\log 3=\cdot 4771213$, find $\log\sqrt{5}$, $\log\sqrt[3]{\cdot 0020736}$, $\log 187\cdot 5$, and $\log\cdot 00288$.

286. Shew that

$$\cos^2 18^0\cos 36^0-\sin 18^0\sin^2 36^0=\frac{5}{8}.$$

287. Shew that

$$\frac{\cos 2A-\cos 5A}{\cos 3A-\cos 4A}=3\cos^2\frac{A}{2}-\sin^2\frac{A}{2}.$$

288. Shew that

$$\tan(A+B)+\tan(A-B) \\ =\tan 2A\,\{1-\tan(A-B)\tan(A+B)\}.$$

289. Solve the equation $\sin 2x=3\tan x\cos 2x$.

290. Shew that

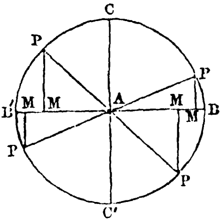
$$\sin 5A-5\sin 3A+10\sin A=16\sin^5 A.$$

291. If in a triangle $\dfrac{13-5\cos A}{13-5\cos B}=\left(\dfrac{\sin A}{\sin B}\right)^2$, then the triangle is either isosceles or such that the sides opposite to A and B are together five times the remaining side.

XVI. *Changes in the Ratios as the angle changes.*

150. In the present Chapter we shall trace the changes in magnitude and sign of the various Trigonometrical Ratios as the angle changes from zero to four right angles.

151. *To trace the changes in the sine of an angle as the angle varies.*



Let BAB' and CAC' be two straight lines at right angles, and suppose a straight line AP of constant length to turn round one end A from the fixed position AB, so that P traces out the circle $BCB'C'$. From any position of P draw PM perpendicular to BAB'; then

$$\sin PAB = \frac{PM}{AP}.$$

When AP coincides with AB the perpendicular PM vanishes; thus when the angle is zero so also is its sine. While AP moves through the first quadrant PM is positive, and continually increases until AP coincides with AC, and then PM is equal to AP; thus as the angle increases from 0 to 90° the sine increases from 0 to 1. While AP moves through the second quadrant PM is positive and continually decreases until AP coincides with AB', and then PM vanishes; thus as the angle increases from 90° to 180° the sine diminishes from 1 to 0. While AP moves through the third quadrant PM is negative and increases *numerically* until AP coincides with AC'; thus as the angle increases from 180° to 270° the sine is *negative* and increases numerically from 0 to -1. While AP moves through the fourth quadrant PM is negative

280. Shew that $\sin\left(\tan^{-1}\frac{3}{4}+\cot^{-1}\frac{5}{12}\right)=\frac{63}{65}$.

281. The angles of a triangle are in the proportion of 2, 3, and 5: express the angles in grades and degrees.

282. Solve the equation

$$\sqrt{3}\tan^2\theta-(1+\sqrt{3})\tan\theta+1=0.$$

283. Given $a=8$, $b=7$, $A=120^0$: find B and C.

$\log 7=\cdot 8450980$, $L\sin 60^0=9\cdot 9375306$,

$\log 8=\cdot 9030900$, $L\sin 49^0\,16'=9\cdot 8795287$,

$L\sin 49^0\,17'=9\cdot 8796375$.

284. Given $b=50$, $c=30$, $A=30^0$: find B and C.

$\log 2=\cdot 3010300$, $L\tan 43^0=9\cdot 9696559$,

$L\cot 15^0=10\cdot 5719475$, $L\tan 43^0\,1'=9\cdot 9699091$.

285. Given $\log 2=\cdot 3010300$, and $\log 3=\cdot 4771213$, find $\log\sqrt{5}$, $\log\sqrt[3]{\cdot 0020736}$, $\log 187\cdot 5$, and $\log\cdot 00288$.

286. Shew that

$$\cos^2 18^0\cos 36^0-\sin 18^0\sin^2 36^0=\frac{5}{8}.$$

287. Shew that

$$\frac{\cos 2A-\cos 5A}{\cos 3A-\cos 4A}=3\cos^2\frac{A}{2}-\sin^2\frac{A}{2}.$$

288. Shew that

$$\tan(A+B)+\tan(A-B)$$
$$=\tan 2A\,\{1-\tan(A-B)\tan(A+B)\}.$$

289. Solve the equation $\sin 2x=3\tan x\cos 2x$.

290. Shew that

$$\sin 5A-5\sin 3A+10\sin A=16\sin^5 A.$$

291. If in a triangle $\frac{13-5\cos A}{13-5\cos B}=\left(\frac{\sin A}{\sin B}\right)^2$, then the triangle is either isosceles or such that the sides opposite to A and B are together five times the remaining side.

292. If a triangle be formed by drawing through the angle A of a plane triangle a straight line parallel to the opposite side, and through the other angles B, C straight lines at right angles respectively to AB, AC, find the sides and the angles of the triangle so formed. Shew that the area of the triangle is

$$\frac{a^2 \cos^2 (B-C)}{2 \sin A \cos B \cos C}.$$

293. Shew that

$$\cos 7A + \cos 5A + \cos 3A + \cos A = 4 \cos 4A \cos 2A \cos A.$$

294. Find all the values of θ less than 180^0 for which $\sin 3\theta = \sin 30^0$.

295. Shew that

$$\sin (A-B) \cos (A+B) + \sin (B-C) \cos (B+C) + \sin (C-A) \cos (C+A) = 0.$$

296. The angles A, B, C of a triangle are in Arithmetical Progression; and $\sin A \sin C = \cos^2 B$: find the angles.

297. Shew that

$$\operatorname{cosec} 2A + \cot 4A + \operatorname{cosec} 4A = \cot A.$$

298. Given $b = 17$, $c = 7$, $A = 60^0$: find B and C.

$\log 2 = \cdot 3010300$, $L \tan 35^0 49' = 9\cdot 8583357$,
$\log 3 = \cdot 4771213$, $L \tan 35^0 49' 10'' = 9\cdot 8583800$.

299. Equilateral triangles DBC, $D'BC$ are described on the side BC of the triangle ABC: shew that

$$AD^2 + AD'^2 = a^2 + b^2 + c^2,$$

and $$AD^2 . AD'^2 = a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2.$$

300. Express $\cos DAD'$ in terms of a, b, c in the preceding Example. Hence shew that if equilateral triangles are in like manner constructed on the other two sides of the original triangle, and the angles DAD', EBE', FCF' be denoted by α, β, γ,

$$\cos \alpha + \cos \beta + \cos \gamma = 0, \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0.$$

ANSWERS.

When an angle is given in degrees, minutes and seconds, it should be brought into degrees and decimals of a degree before expressing it in the French mode. Thus, for example, $1^0\,21' = 1^0{\cdot}35$.

I. 1. 60^g. 2. $1^g{\cdot}5$. 3. 7^g. 4. $10^g{\cdot}925$.
5. $30^g{\cdot}775$. 6. $74^g{\cdot}925$. 7. 27^0. 8. $3^0\,9'$.
9. $9^0\,22'\,57''$. 10. $18^0\,41'\,51''$. 11. $27^0\,58'\,3''$.
12. $68^0{\cdot}8068$. 13. 18^0, 20^g. 14. $56^0\,15'$, $62^g{\cdot}5$.
15. $61^0{\cdot}875$, $68^g{\cdot}75$. 16. $63^0{\cdot}28125$, $70^g{\cdot}3125$. 17. 60^0, $66^g\tfrac{2}{3}$.
18. 36^0, 40^g. 19. 18^0, 9^0. 20. 54^0, 36^0. 21. 54.
22. 32·4. 23. $111\tfrac{1}{9}$. 24. $185\tfrac{5}{27}$. 25. 34 to 27.

II. 1. $\cos A = \frac{5}{13}$. 2. $\cos A = \frac{9}{41}$. 3. $\sin A = \frac{24}{25}$.
4. $\sin A = \frac{11}{61}$. 5. $\sin A = \frac{4}{5}$. 6. $\cos A = \frac{2\sqrt{2}}{3}$.
7. $\sin A = \frac{\sqrt{7}}{4}$. 8. $\cos A = \frac{m^2-1}{m^2+1}$. 9. $\sin A = \frac{m^2-n^2}{m^2+n^2}$.

III. 1. 30^0. 2. 60^0. 3. 60^0. 4. 60^0.
5. 45^0 or 60^0. 6. 15^0 or 75^0. 7. $A = 45^0$, $B = 30^0$.
8. $A = 60^0$, $B = 45^0$. 9. $A = 45^0$, $B = 15^0$.
10. $A = 18^0$, $B = 24^0$. 11. $A = 10^0$, $B = 5^0$.
12. $A = 45^0$, $B = 45^0$. 13. $A = 15^0$, $B = 30^0$, $C = 15^0$.

14. $\sin 22\tfrac{1}{2}^0 = \sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}$, $\cos 22\tfrac{1}{2}^0 = \sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}$,

$\tan 22\tfrac{1}{2}^0 = \sqrt{\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)}$; it may be reduced to $\sqrt{2}-1$.

$\cot 22\tfrac{1}{2}^0 = \frac{1}{\tan 22\frac{1}{2}^0} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$.

15. $\frac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$; it may be reduced to $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

16. $\frac{\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}$; it may be reduced to $\sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

IV. 1. $\sin A = \frac{21}{29}$, $\cos A = \frac{20}{29}$.

2. $\cos A = \frac{m(2n+m)}{2n^2+2nm+m^2}$. 3. $\sin A = \frac{pm+qn}{\sqrt{(p^2+q^2)}\sqrt{(m^2+n^2)}}$, $\cos A = \frac{pn-qm}{\sqrt{(p^2+q^2)}\sqrt{(m^2+n^2)}}$. 4. $\tan x = a \pm \sqrt{(a^2-1)}$.

5. $\sin x = \frac{1}{2}\{a \pm \sqrt{(2-a^2)}\}$. 6. $A = 15^0$ or 75^0.

7. $A = 30^0$ or 60^0, $B = 60^0$ or 30^0.

8. $A = 15^0$ or 75^0, $B = 75^0$ or 15^0. 9. $A = 15^0$ or 75^0, $B = 75^0$ or 15^0. 11. $\frac{100}{\sqrt{3}}$ feet. 12. $50\,(3-\sqrt{3})$ yards.

13. $10\,(3+2\sqrt{3})$ feet, $10\sqrt{3}$ feet. 14. $\frac{a^2\sqrt{3}}{4}$.

15. $\frac{a^2}{4}$. 16. 27 feet. 17. 1 or 2: assume $2h$ for the height of the post, then we shall find that the distance of the point of observation is either h or $2h$.

18. $(a+b)(2-\sqrt{3})$. 19. 100 feet.

20. $\frac{h}{h+70} = \frac{1}{3}$ gives $h = 35$; $35 + 5 = 40$.

V. 1. 8. 2. $\frac{5}{2}$. 3. $\frac{5}{2}$. 4. $\frac{4}{3}$. 5. $\frac{3}{2}$. 6. $\frac{7}{5}$.

7. 1·2552726. 8. 1·7781513. 9. 2·3344539.
10. 3·8115752. 11. 3·7323939. 12. $\bar{1}$·6478174.
13. ·6354839. 14. $\bar{1}$·8573326. 15. $\bar{1}$·5740313.
16. $\bar{2}$·4771213. 17. $\bar{1}$·7406162. 18. $\bar{1}$·6365006.
19. 1·7993406. 20. $\bar{1}$·2640466. 21. $\bar{3}$·3555614.
22. ·0511525. 23. ·0880456. 24. ·0263938.
25. ·3521825. 26. 0, 2, $\bar{2}$, $\bar{5}$; $\bar{3}$·1518804.
27. $\bar{1}$·7041854. 28. $-\frac{4}{3}$. 29. 4·4983106. Observe $315 = \frac{14 \times 15 \times 15}{10}$. 30. $\frac{3}{2-\log.2} = 1{\cdot}7657757$.

31. $\dfrac{2-\log 2}{2-2\log 2}=1{\cdot}2153383.$

32. $\dfrac{3}{3\log 3+2\log 2-2}=89{\cdot}75613.$

VI. 1. 2·5389050. 2. 6·6180633. 3. 2·8663518.
4. 1·2435557. 5. 2·7855128. 6. 6·7915179.
7. 616·0149. 8. ·007501467. 9. 2.
10. 2·6533. 11. 9·7932666. 12. 9·7299388.
13. 9·8241849. 14. 9·9793629. 15. 9·5906364.
16. 9·7154581. 17. 10·1650011. 18. 10·2618877.
19. 10·0171747, 9·4577109. 20. $16^0\ 19'\ 26''$.
21. $6^0\ 53'\ 8''$. 22. $22^0\ 28'\ 16''$.
23. $80^0\ 52'\ 51''$. 24. 142·7035.

VII. 1. $a=75,\ b=75\sqrt{3}$. 2. $A=30^0,\ b=100\sqrt{3}$.
3. $b=80\,(2-\sqrt{3}),\ c=80\,(\sqrt{6}-\sqrt{2})$.
4. $A=45^0,\ c=75\sqrt{2}$. 5. $a=97{\cdot}082040,\ b=70{\cdot}534236$.
6. $A=16^0\ 15'\ 37'',\ b=24$. 7. $A=43^0\ 36'\ 10'',\ b=210$.
8. $b=125\,(\sqrt{2}-1),\ c=125\sqrt{(4-2\sqrt{2})}$.
9. $A=46^0\ 23'\ 59'',\ c=29$. 10. $c=5,\ B=53^0\ 7'\ 48''$.
11. $a=78{\cdot}1548,\ b=179{\cdot}744$.
12. $A=35^0\ 49'\ 44'',\ b=132{\cdot}966$.
13. $b=59{\cdot}87667,\ c=138{\cdot}24$.
14. $A=36^0\ 9'\ 3'',\ c=239{\cdot}02$.

VIII. 1. $b=84\sqrt{2},\ a=42\,(\sqrt{2}+\sqrt{6})$.
2. $C=30^0,\ a=48\sqrt{3}$.
3. $B=45^0$ or $135^0,\ c=\dfrac{1}{2}(\sqrt{6}+3\sqrt{2})$ or $\dfrac{1}{2}(\sqrt{6}+\sqrt{2})$.
4. $B=90^0, c=2\sqrt{2}-\sqrt{6}$. 5. $B=30^0$. 6. $A=90^0, B=60^0$.
7. $a=107{\cdot}087,\ b=96{\cdot}5836$. 8. $c=326{\cdot}576$.
9. $B=55^0\ 13'\ 28''$ or $124^0\ 46'\ 32'',\ c=265{\cdot}343$ or $87{\cdot}389$.
10. $B=55^0\ 9'\ 8'',\ c=537{\cdot}079$.
11. $B=40^0\ 56'\ 38{\cdot}5'',\ c=106{\cdot}736$.
12. $A=38^0\ 25'\ 19'',\ B=57^0\ 41'\ 24''$. 15. 60.

IX. 1. 108, 120. 2. 75 grades. 3. 30^0.

4. $\tan\frac{1}{2}A=\frac{1}{5}$. 6. $\cos^2 A=\frac{1}{4}(2-\sqrt{2})$, $\cos 2A=-\frac{1}{\sqrt{2}}$.

8. $\cos 2A=-\frac{4}{5}$, $\cos A=\frac{1}{\sqrt{10}}$.

10. $\sin^2 A=\frac{m^2(1-n^2)}{m^2-n^2}$, $\sin^2 B=\frac{1-n^2}{m^2-n^2}$.

11. $\sin^2 A=\frac{m^2-n^2}{1-n^2}$, $\sin^2 B=\frac{m^2-n^2}{m^2(1-n^2)}$; also we may have $\sin A=0$, $\sin B=0$. 14. $\sin A=\frac{40}{41}$.

16. $\sin^2 x=\frac{mn-mb-na}{(m-n)(a-b)}$, $\sin^2 y=\frac{m-a-b}{m-n}$.

21. $B=75^0\ 57'\ 50''$. 22. $A=38^0\ 15'\ 8''$.

24. $\pm\left\{a-\sqrt{\left(\frac{1}{2}-a^2\right)}\right\}$.

X. 1. 30^0 or 150^0. 3. Here it will be found that $\frac{b-c}{b+c}=\frac{2-\sqrt{3}}{\sqrt{3}}$; and $\cot 15^0$ is given in Art. 31. We get $C=45^0$. 4. $B=90^0$, $c=a\sqrt{3}$. 5. 120^0.
6. 90^0. 7. 120^0. 8. 14, 12. 10. 4·58257.
14. $\pm\{pq-\sqrt{(1-p^2)(1-q^2)}\}$. 17. 30^0 or 150^0.
19. $\frac{m\sin a}{\sin(\beta-a)}$. 20. Substitute in the third equation the values of $\tan^2 x$ and $\tan^2 y$ from the first two; see Example IX. 16: then reduce the result.

XI. 1. 226·623. 2. 71·0919. 3. 1239·632.
4 $B=149^0\ 20'\ 31''$, $C=24^0\ 39'\ 29''$.
5. $B=116^0\ 33'\ 54''$, $C=26^0\ 33'\ 54''$.
6 $B=76^0\ 44'\ 55''$, $C=53^0\ 59'\ 5''$.
7. $B=101^0\ 29'\ 9''$, $C=36^0\ 0'\ 51''$.
8. $B=86^0\ 34'\ 27''$, $C=50^0\ 15'\ 33''$.

9. $B=68^{\circ}\,2'\,24''$, $C=54^{\circ}\,4'\,19''{\cdot}2$.

10. $B=81^{\circ}\,2'\,16''$, $C=50^{\circ}\,37'\,44''$, $a=4{\cdot}3485$.

11. $B=23^{\circ}\,8'\,33''$, $C=32^{\circ}\,17'\,27''$.

12. $B=35^{\circ}\,15'\,52''$, $C=84^{\circ}\,44'\,8''$, $c=137{\cdot}9796$.

13. $B=17^{\circ}\,6'\,45''$, $C=133^{\circ}\,2'\,15''$.

14. $B=51^{\circ}\,41'\,20''$, $C=58^{\circ}\,6'\,30''$, $\log c={\cdot}9271876$.

15. $B=41^{\circ}\,19'\,20''$ or $138^{\circ}\,40'\,40''$,
$C=105^{\circ}\,58'\,40''$ or $8^{\circ}\,37'\,20''$.

16. $B=30^{\circ}$ or 150°.

17. $A=32^{\circ}\,57'\,8''$.

18. $43^{\circ}\,2'\,56''$, $53^{\circ}\,46'\,44''$, $83^{\circ}\,10'\,20''$.

19. $56^{\circ}\,15'\,4''$, $59^{\circ}\,51'\,10''$, $63^{\circ}\,53'\,46''$.

20. $49^{\circ}\,6'\,24''$, 60°, $70^{\circ}\,53'\,36''$.

XII. 1. 72 feet. 2. $\dfrac{\sqrt{3}+1}{\sqrt{2}}$ miles.

3. $500\,(\sqrt{3}+1)$ yards. 4. $18\sqrt{2}$ miles.

5. 93·489 yards. 6. 155·823 feet.

7. 228·6307 yards. 8. 280·015, 765·015 feet.

9. 212·858, 618·186 feet. 10. E $7^{\circ}\,21'\,45''$ S.

11. 238·502, 480·504 yards. 12. $4\sin 67\tfrac{1}{2}^{\circ} - 4\sin 22\tfrac{1}{2}^{\circ}$.

13. Distance $=\dfrac{3\sin 45^{\circ}}{\sin 22\frac{1}{2}^{\circ}}$; the first ship bears E.N.E. of the second. 14. $40\sqrt{330}$ feet per minute.

15. Height 80 feet; distance $4800\sqrt{330}$ feet.

16. If h be the height $3\,(200)^2=h^2\,(4-\sqrt{6})$.

18. Solve *CEF* knowing *EF* and the angles; then solve *AEC* and *BFC*.

20. The straight line drawn from the sun to the eye, and the straight line drawn from the sun to the shadow of the cloud, may both be supposed inclined at an angle β to the horizon, on account of the great distance of the sun.

22. The balloon is moving towards the W.N.W., at the rate of $\dfrac{1}{2}\,(3+\sqrt{3})$ miles an hour.

23. 162·95. 24. 1514·396, 4163·746. 25. 509·77.

26. 2109·8. 27. $SA=777$, $SB=502$, $SC=790$.

XIII. 1. 72^0. 2. $36^0, 54^0$. 8. $\alpha=45^0, \beta=30^0$.

12. $10(\sqrt{2}-1)$, $10\sqrt{4-2\sqrt{2}}$.

13. $AC=\dfrac{h\cos\beta\cos\alpha}{\sin(\alpha-\beta)}$, $CD=\dfrac{h\cos\beta\sin\alpha}{\sin(\alpha-\beta)}$.

14. $\dfrac{C}{2}=41^0\,24'\,35''$, $B=\dfrac{C}{2}$: see Example x. 9.

15. $\dfrac{C}{2}=57^0\,41'\,18''$, $B=\dfrac{C}{2}$.

16. A circle will go round $ABCD$.

18. $\sin^2 A=\dfrac{c(b-c)}{a(a+b-2c)}$, $\sin^2 B=\dfrac{c(a-c)}{b(a+b-2c)}$.

XIV. 3. $\dfrac{a}{2\sqrt{3}}$. 4. $\dfrac{a\sqrt{3}}{2}$. 5. $\dfrac{a}{\sqrt{3}}$.

6. $S=2310$, $r=21$, $R=42\frac{1}{2}$. 7. $6\times(81)^2$.

8. 2456840. 9. 382094.

18. The radius of the circle passing through the three points is $\dfrac{41\times 13}{8}$; the tower is at the centre of this circle.

XV. 1. $30^0, 150^0, 210^0, 330^0$. 2. $45^0, 135^0, 225^0, 315^0$.

3. $60^0, 120^0, 240^0, 300^0$. 4. $30^0, 60^0, 210^0, 240^0$.

5. 60^0 $120^0, 240^0, 300^0$. 6. $22\frac{1}{2}^0, 112\frac{1}{2}^0, 202\frac{1}{2}^0, 292\frac{1}{2}^0$.

7. $90^0, 210^0, 330^0$. 8. $120^0, 180^0, 240^0$.

9. $15^0, 75^0, 195^0, 255^0$. 10. $67\frac{1}{2}^0, 157\frac{1}{2}^0, 247\frac{1}{2}^0, 337\frac{1}{2}^0$.

11. $0^0, 30^0, 150^0, 180^0$. 12. $60^0, 90^0, 270^0, 300^0$.

XVI. 1. In the first quadrant from 1 to $\sqrt{2}$, and then from $\sqrt{2}$ to 1; in the second quadrant from 1 to 0, and then from 0 to -1; in the third quadrant from -1 to $-\sqrt{2}$, and then from $-\sqrt{2}$ to -1; in the fourth quadrant from -1 to 0, and then from 0 to 1. 2. In the first quadrant from -1 to 0; and then from 0 to 1; in the second quadrant from 1 to $\sqrt{2}$, and then from $\sqrt{2}$ to 1; in the third quadrant from 1 to 0, and then from 0 to -1; in the fourth quadrant from -1 to $-\sqrt{2}$, and then from $-\sqrt{2}$ to -1. 3. In the first quadrant from 0 to 1; in the second quadrant from 1 to 0; in the third and fourth quadrants as in the first and second respectively. 4. In the first quadrant from 1 to 0, and then from 0 to -1; in the second quadrant from -1 to 0, and then from 0 to 1; in the third and fourth quadrants as in the first and second respectively. 5. In the first quadrant from infinity to 2; in the second quadrant from 2 to infinity; in the third quadrant from $-$ infinity to -2; in the fourth quadrant from -2 to $-$ infinity. 6. In the first quadrant from 1 to infinity; in the second quadrant from $-$ infinity to -1; in the third quadrant from -1 to 0; in the fourth quadrant from 0 to 1.

7. $\tan^2 A + \cot^2 A = (\tan A - \cot A)^2 + 2$. 8. 4.

11. $12150 + 6250\sqrt{2}$.

XVII. 1. $-\dfrac{1}{\sqrt{2}}$. 2. 1. 3. $\dfrac{\sqrt{3}}{2}$. 4. $-\dfrac{\sqrt{3}}{2}$.

5. -1. 6. $\dfrac{1}{2}$. 7. $2-\sqrt{3}$. 8. $-(2-\sqrt{3})$.

9. $\sqrt{3}$. 10. $\dfrac{1}{\sqrt{3}}$. 11. $-\sqrt{3}$. 12. 1.

XVIII. 7. $A = (4n+1)\,90^\circ$. 8. $90^\circ - 2A = n360^\circ \pm 3A$.

XIX. 1. -1. 2. $\dfrac{204}{325}$. 3. $\dfrac{2499}{2501}$. 4. $\dfrac{\sqrt{3}+\sqrt{15}}{8}$.

6. $\dfrac{(1-ab)^2-(a+b)^2}{(1+a^2)(1+b^2)}$, $\dfrac{2(a+b)(1-ab)}{(1+a^2)(1+b^2)}$. 8. $-\dfrac{1}{2}$.

9. $\dfrac{3}{5}$ or $-\dfrac{4}{5}$. 11. $\dfrac{p^2(2-c^2)-q^2}{q^2-p^2c^2}$ or $\dfrac{p^2c^2+q^2(1-2c^2)}{q^2-p^2c^2}$. 13. 74°.

XX. 27 may be deduced from 26; 28 may be deduced from 26 by changing angles into their complements; 29 may be deduced from 28; 30 may be deduced from 26 by changing C into $90^\circ - C$.

36. $x = n\,180^\circ$, or $x = n\,180^\circ + (-1)^n\,30^\circ$.

37. $4x = x + n180^\circ$.

38. $2x=(2n+1)\,90^0$, or $7x=n180^0+(-1)^n\,30^0$.

39. $8x=n180^0$, or $8x=n360^0\pm 60^0$.

40. $(a+b+2c)\,x=n360^0\pm(a+b)\,x$. 41. $\sqrt{3}$.

45. $1\pm\sqrt{\dfrac{2b}{1+b}}$.

XXI. 1. $\dfrac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2}$. 3. $\dfrac{13+\sqrt{31}}{20}$.

XXII. 7. $\dfrac{360}{\pi}$ feet, $\dfrac{54}{10}$ feet. 8. $\dfrac{126}{\pi}$.

9. $\dfrac{10\pi}{3}$ feet. 10. $50^0, 40^0$. 11. $40^0, 60^0, 80^0$.

12. $\left(m-\dfrac{9}{10}n\right)\dfrac{\pi}{180}$. 13. $\dfrac{50}{27}\,\dfrac{60a+b}{100a+b}$. 15. $1\frac{8}{10}^0$.

16. $\dfrac{10a+9b}{900}$. 19. $\sin^2\theta=\dfrac{\sin^2 a-k\cos^2 a}{1-k}$.

XXIII. 1. $\dfrac{a^2}{4}\left(\sqrt{3}-\dfrac{\pi}{2}\right)$, where a is a side of the triangle. 2. The smaller segment is $\dfrac{\pi r^2}{6}-\dfrac{r^2\sqrt{3}}{4}$.

3. $R^2\left(\dfrac{\pi}{4}-\dfrac{1}{2}\right)$, $R^2\left(\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{4}\right)$, $R^2\left(\dfrac{5\pi}{12}-\dfrac{1}{4}\right)$.

6. We must have $\theta=\cos\theta$, and therefore $1-\dfrac{\theta^2}{2}$ less than θ.

8. $42^0\,50'\,22''$, $17^0\,9'\,38''$. 9. $71^0\,5'\,45''$, $48^0\,54'\,15''$.

10. $83^0\,58'\,28''$.

11. Let x be the height of the lower; then that of the other is $2x$; and $4x^2=(3b-2a)\,(2a-b)$.

12. 7·11423 miles.

XXIV. 4. $n180^0+(-1)^n\,90^0$, this may be simplified to $(4m+1)\,90^0$; $\tan\{n\,180^0+(-1)^n\,45^0\}$, this may be reduced to $(-1)^n$. 10. $x=1$ or $\dfrac{1}{3}$.

MISCELLANEOUS EXAMPLES.

1. 27°, 30°. 2. $\dfrac{a}{\sqrt{(a^2+b^2)}}, \dfrac{b}{\sqrt{(a^2+b^2)}}$. 4. $2A = 30°$.
5. $2A = 45°$. 6. $\dfrac{5921}{20000}$. 7. As $\sqrt{3}$ is to 2. 9. $\dfrac{5}{4}, \dfrac{4}{5}$.
10. 4·8303525. 11. $4\frac{1}{2}$. 12. 60°, 45°, 135°, 120°.
15. $A = 90°$. 16. $2A = 45°$. 18. $40\sqrt{3}+5$ feet. 19. 3, 2.
20. 400, 420. 21. 27°·414. 22. 72°, 80°. 25. $2A = 45°$.
26. $A = 30°$ or 60°. 28. $\dfrac{5}{12}$. 29. 1·3979400, $\bar{1}$·0969100.
30. 2895·256. 31. 22°·2048. 34. $2A = 60°$. 35. $2A = 45°$.
37. 60°. 39. ·9030900, $\bar{2}$·8061800. 40. 12; $\dfrac{12}{37}$. 44. Unity.
45. $4A = 60°$. 46. Find a from the second equation, substitute in the first, and solve the quadratic in $\dfrac{x^3}{y^3}$.
48. If c be the difference of the lengths of the shadows $c = h(\cot\alpha - \cot\beta)$. In the second part of the Example $\alpha+\beta = 90°$. 49. 3, $\dfrac{1}{9}$, -2. 50. 9·4918055. 51. $82\frac{1}{2}°$, 60°, $37\frac{1}{2}°$. 52. $\left(\dfrac{x}{a}\right)^{\frac{2}{m}} + \left(\dfrac{y}{b}\right)^{\frac{2}{m}} + \left(\dfrac{z}{c}\right)^{\frac{2}{m}} = 1$. 53. $x = 10°$.
54. $\dfrac{4}{5}$. 55. 30 feet. 59. ·4771212, 1·8750612, $\bar{2}$·8750612.
60. 26° 39′ 23″. 61. 54°, 54°, 72° or 66°, 66°, 48°. 62. $\dfrac{1}{2}$.
65. $2A = 90°$. 67. As 1, $\sqrt{3}$, and 2. 69. 3·2375439, 1·2375439, $\bar{3}$·2375439. 70. 9·6312546. 71. 36°, 54°.
72. $\sin^2 A = \dfrac{q^2(1-p^2)}{q^2-1}$, $\sin^2 B = \dfrac{1-p^2}{p^2(q^2-1)}$. 75. $-1+\sqrt{2}$.
78. $\dfrac{c\sin\alpha}{2\cos\beta}$. 79. $\bar{1}$·9262139, $\bar{3}$·7269987. 80. 70° 40′ 17″.

81. $39^{0}\cdot5505$. 82. $\sin A = \frac{1}{2}\left(p - \frac{q}{p}\right)$, $\sin B = \frac{1}{2}\left(p + \frac{q}{p}\right)$.

86. 50 yards. 88. $2ab$. 89. 1·5185140, 3·0124153, $\bar{1}\cdot1212040$. 90. 9·5285910. 91. $\frac{5}{6}$. 92. a. 95. $\pm\frac{1}{\sqrt{2}}$.

97. $2ab$ if $\frac{a}{b}$ is not greater than unity; a^2+b^2 if $\frac{a}{b}$ is greater than unity. 99. 1·4434112. 100. $59^{0}\,20'\,33''$.

103. $2A = 30^{0}$ or 150^{0}. 104. 1·4313639, 1·5563026, 1·7323939, $\bar{3}\cdot3979400$, $\bar{1}\cdot0915148$. 105. 12 square inches.

106. 150 feet. 107. $B = 76^{0}\,47'\,2''$, $C = 49^{0}\,12'\,58''$.

112. 3. 113. 1 or $\frac{3}{5}$. 114. $6\sqrt{10}$. 115. 50 feet.

116. $B = 66^{0}\,14'\,38''$, $C = 50^{0}\,45'\,22''$. 119. 120^{0}.

121. 25^{g}, 44^{g}, $84^{g}\cdot9$. 123. $2A = 90^{0}$ or 180^{0}.

124. $\frac{1\pm\sqrt{7}}{4}$. 125. $\frac{11}{14}$, $\frac{1}{2}$, $\frac{1}{7}$; $\frac{7\sqrt{3}}{3}$. 126. $10\sqrt{3}$ feet.

130. 1469·69 square yards. 132. $\frac{2}{3}$. 133. $\frac{1\pm\sqrt{5}}{4}$. 134. $\frac{5}{8}$.

136. ·7071068, ·924, ·383. 139. $B = 168^{0}\,27'\,25''\cdot4$, $C = 4^{0}\,55'\,10''\cdot6$. 142. $2\sin A = p \pm r$, $2\sin B = p \mp r$; where $r^2 = \frac{4q - p^3}{3p}$. 143. $\frac{ap - bq}{aq + bp}$. 148. $\sqrt{(2ch\cot a - c^2 + h^2)} - h$.

151. $2A = 90^{0}$. 152. $\bar{1}\cdot8785218$, $\bar{1}\cdot6261739$. 153. $B = 75^{0}$, $C = 90^{0}$, $c = 2\sqrt{2}$; or $B = 105^{0}$, $C = 60^{0}$, $c = \sqrt{6}$.

154. 172·6436. 155. $A = 80^{0}\,30'$, $C = A$.

157. $\tan^2\theta = 3$ or $\frac{1}{3}$. 159. $\frac{c\sqrt{(4r^2 - c^2)}}{2r^2 - c^2}$.

161. $a^2 + b^2 = h^2 + k^2$. 165. $\frac{2ab}{b^2 - a^2}$. 166. $55^{0}\,46'\,16''$.

167. $B = 71^0\,44'\,30''$, $C = 48^0\,15'\,30''$. 170. $A = 45^0$, $B = 30^0$, $C = 15^0$. 171. $2A = 30^0$ or 270^0. 172. 1·06.

173. $\sin x = \frac{\sqrt{5}-1}{2}$. 174. $A = 34^0\,27'$, $B = 100^0\,33'$, $b = 347{\cdot}5767$. 175. $A = 117^0\,38'\,45''$, $B = 27^0\,38'\,45''$.

176. $70^0\,53'\,36''$, $49^0\,6'\,24''$. 177. $\frac{2bc \sin A}{b^2 - c^2}$.

181. ·6020600, ·5440680, $\overline{3}{\cdot}1461280$, $\overline{1}{\cdot}3502480$.

182. $\frac{2ab}{b^2 - a^2}$. 183. $2A = 30^0$. 184. 120^0, $\frac{15\sqrt{3}}{4}$.

186. $A = 106^0 20' 12''$, $B = 13^0 39' 48''$. 190. It will be found that P, A, and R are on a straight line; that $PAQ = 90^0$, and $BAR = 60^0$; also $AP = AB$, $AS = AB$, $AR = 2AB$, $AQ = \sqrt{3}\,.\,AB$. 191. $\frac{1}{6}$. 192. 407·604.

193. 95523·5 sq. yards. 194. $A = \pm 30^0$. 195. $c \tan \alpha \sqrt{2 - \sqrt{2}}$.

196. $(a'b + cb')^2 + (ab' + c'b)^2 = (cc' - aa')^2$. 205. $7\frac{1}{2}^0$, $37\frac{1}{2}^0$, $97\frac{1}{2}^0$, $127\frac{1}{2}^0$. 207. $54^0\,46'\,51''$. 208. $3\theta = (2n + 1)\,90^0$, or $2\theta = n\,180^0 + (-1)^n 30^0$. 215. $4\theta = n 180^0 + (-1)^n \theta$.

216. $\frac{\pi r}{3}$. 217. $\theta = n\,180^0$ or $4\theta = n\,180^0 + (-1)^n\,30^0$.

226. 25·7. 227. $\frac{\pi}{25}$. 232. $x = (3n + 1)\,30^0$. 237. ·913.

246. $173\frac{1}{3}^0$. 247. $3a^2 + b^2 + 2\sqrt{3}S$, $3b^2 + a^2 + 2\sqrt{3}S$.

251. $25^g{\cdot}225$, $72^g{\cdot}725$. 252. 2·5105452, $\overline{2}{\cdot}1760913$. 253. $\sqrt{3}$, 1·5.

254. 156·0412. 255. $88\,(\sqrt{3} + 1)$. 258. 12·85....

259. $x = n 90^0$. 261. $\pm n$. 262. $65^0\,37'\,41''$, $29^0\,22'\,19''$.

263. $A = 15^0$, $a = \sqrt{30}\,(\sqrt{3} - 1)$. 268. $x - 45^0 = n 360^0 \pm 45^0$.

269. $7^0{\cdot}16\ldots$ 271. $\frac{1^c}{10}$, $\frac{1^g}{9}$, $\frac{\pi}{1200}$. 272. 90^0, 72^0.

273. $21^\circ\,47'\,12''$. 274. $A = 103^\circ\,39'\,48''$, $B = 16^\circ\,20'\,12''$.

278. $x = n180^\circ + 45^\circ$, or $x - 45^\circ = n360^\circ \pm 135^\circ$.

279. 154, 9240. 281. $36^\circ, 54^\circ, 90^\circ$; $40^g, 60^g, 100^g$.

282. $\theta = n180^\circ + 45^\circ$ or $n180^\circ + 30^\circ$. 283. $B = 49^\circ\,16'\,5''$, $C = 10^\circ\,43'\,55''$. 284. $B = 118^\circ\,0'\,55''$, $C = 31^\circ\,59'\,5''$.

285. $\cdot 3494850$, $\bar{1}\cdot 1055751$, $2\cdot 2730013$, $\bar{3}\cdot 4593926$.

289. $\sin x = 0$ or $\cos 2x = \frac{1}{2}$. 294. $10^\circ, 50^\circ, 130^\circ, 170^\circ$

296. $15^\circ, 60^\circ, 105^\circ$. 298. $B = 95^\circ\,49'\,3''$, $C = 24^\circ\,10'\,57''$.

THE END.

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